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Formal differential operators, vertex operator algebras and zeta-values, II

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Abstract

We introduce certain correlation functions (graded q-traces) associated to vertex operator algebras and superalgebras which we refer to as n-point functions. These naturally arise in the studies of representations of Lie algebras of differential operators on the circle (J. Lepowsky, to appear, J. Lepowsky, In: Recent Developments in Quantum Affine Algebras and Related Topics (Raleigh, NC, 1998), Amer. Math. Soc., Providence, RI, 1999, p. 327, A. Milas, Formal Differential Operators, Vertex Operator Algebras and Zeta-values, I, to appear). We investigate their properties and consider the corresponding graded q-traces in parallel with the passage from genus 0 to genus 1 conformal field theory. By using the vertex operator algebra theory we analyze in detail correlation functions in some particular cases. We obtain elliptic transformation properties for q-traces and the corresponding q-difference equations. In particular, our construction leads to certain correlation functions and q-difference equations investigated by Bloch and Okounkov (Adv. Math. 149 (2000), 1).

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1. Introduction

This is a continuation of [24]. In this part we study certain correlation functions built up from the iterates of vertex operators introduced in Part I [24].

Let V be an arbitrary vertex operator (super)algebra and M a V-module. Sometimes we will weakened this property by assuming that M is only a subspace of a V-module invariant with respect to certain operators. Let $u_i \in V$, $i = 1, \ldots, n$, $w \in M$ and $w' \in M'$,

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where M' is the (restricted) dual space of M as defined in [9]. Suppose that x_i's are commuting formal variables as in Part I [24]. In the vertex operator algebra theory one usually studies the following formal *matrix coefficients*

$$\langle w', Y(u_1,x_1)\cdots Y(u_n,x_n)w\rangle, \tag{1.1}$$

and the corresponding graded q-traces (cf. [26])

$$\mathrm{tr}|_M X(u_1,x_1)\cdots X(u_n,x_n)q^{\bar{L}(0)}. \tag{1.2}$$

Matrix coefficients of the type (1.1) are studied in [10] in both the formal and the analytic context (meaning that x_i are set to be complex variables). In particular in [11] expressions of the form (1.1) were used for construction of the genus zero meromorphic conformal field theory. On the other hand the graded traces function of the form (1.2) are related to the genus one meromorphic conformal field theory. More precisely, the expression of the form (1.2) give a vector in a genus-one conformal block associated to V [26].

In Part I [24] we have studied a relationship between various normal orderings and classical Lie subalgebras of the Lie algebra of (super)differential operators on the circle. The most interesting representations of $\hat{\mathscr{D}}$ (unitary representations for instance) can be constructed by using certain quadratic operators in terms of free fields [1,8,17]. As we already noticed, these quadratic operators are closely related to iterates $X(Y[u,y]v,x)$ (cf. [20,21,24]). Therefore it is very natural to consider matrix coefficients of the form

$$\langle w', X(Y[u_1,y_1]v_1,x_1)\cdots X(Y[u_n,y_n]v_n,x_n)w\rangle, \tag{1.3}$$

which we will call iterated $2n$-point function.[1] Also we consider a $2n$-point function:

$$\langle u'_{n+1}, X(u_1,x_1t_1)X(v_1,x_1)\cdots X(u_n,x_nt_n)X(v_n,x_n)u_{n+1}\rangle. \tag{1.4}$$

The first result in our paper is Theorem 2.1 which relates the matrix coefficients (1.3) and (1.4). In parallel with (1.1) and (1.2) it is natural to consider the corresponding graded q-traces:

$$\mathrm{tr}|_M X(Y[u_1,y_1]v_1,x_1)\cdots X(Y[u_n,y_n]v_n,x_n)q^{\bar{L}(0)}, \tag{1.5}$$

Here $\bar{L}(0)=L(0)-c/24$. Again, (cf. Proposition 2.2) (1.5) is closely related to

$$\mathrm{tr}|_M X(u_1,x_1t_1)X(v_1,x_1)\cdots X(u_n,x_nt_n)X(v_n,x_n)q^{\bar{L}(0)}. \tag{1.6}$$

Besides the q-traces that we already mentioned, one would like to consider (for reasons that will become clear in a moment) an expansion of the q-trace (1.6) in powers of the x_i's:

$$\begin{aligned}&\mathrm{tr}|_M X(u_1,x_1t_1)X(v_1,x_1)\cdots X(u_n,x_nt_n)Y(v_n,x_n)q^{\bar{L}(0)}\\&\quad=\sum_{\alpha}(\mathrm{tr}|_M X(u_1,x_1t_1)X(v_1,x_1)\cdots X(u_n,x_nt_n)X(v_n,x_n)q^{L(0)})_\alpha x^\alpha\end{aligned}$$

[1] $2n$ refers to the number of formal (or complex) variables.

(here we use the multi-index notation $x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$). Especially interesting is the *constant term* with respect to the x-variables, i.e.,

$$\begin{aligned}&\mathrm{Coeff}_{x_1^0 \ldots x_n^0} \operatorname{tr}|_M X(u_1, x_1 t_1) X(v_1, x_1) \cdots X(u_n, x_n t_n) X(v_n, x_n) q^{\bar{L}(0)} \\ &\quad = \operatorname{tr}|_M \mathrm{o}(X(u_1, x_1 t_1) X(v_1, x_1)) \cdots \mathrm{o}(X(u_n, x_n t_n) X(v_n, x_n)) q^{\bar{L}(0)} \qquad (1.7)\end{aligned}$$

where $\mathrm{o}(a) = a(\mathrm{wt}(a) - 1)$ (cf. [26]). This new formal expression depends on n-parameters: $t_1, \ldots, t_n$. Also we are interested in

$$\begin{aligned}&\mathrm{Coeff}_{x_1^0 \ldots x_n^0} \operatorname{tr}|_M X(Y[u_1, y_1] v_1, x_1) \cdots X(Y[u_n, y_n] v_n, x_n) q^{\bar{L}(0)} \\ &\quad = \operatorname{tr}|_M \mathrm{o}(Y[u_1, y_1] v_1) \cdots \mathrm{o}(Y[u_n, y_n] v_n) q^{\bar{L}(0)}. \qquad (1.8)\end{aligned}$$

Any expression of the form (1.7) or (1.8) (possibly normalized by $\operatorname{tr}|_M q^{L(0)}$) we shall refer to as *n-point correlation function* (or simply, n-point function). This term is widely used in statistical physics and random matrix theory where an n-point correlation function depending on $y_1, \ldots, y_n$ (or $t_1, \ldots, t_n$) can be defined, for example, as an integral (usually normalized by the partition function) of a probability density function depending on integration variables $x_1, \ldots, x_n$. Because extracting the zeroth term in (1.7) resembles (complex) integration in variables $x_1, \ldots, x_n$ we decided to use this terminology (cf. [3]).

In our applications M will be a V-module stable under the Fourier modes of $X(u_i, xt) X(v_i, x)$. In the case of rational vertex operator algebras with some additional properties (cf. [26]), the q-trace (1.2) rise to a doubly periodic functions (in this case even more is true; the vector space of all characters of V-modules is invariant under $SL(2, \mathbb{Z})$ action). However, to prove ellipticity one needs much weaker assumptions.

Here is a short overview:

(i) In Section 2 we derive a precise relationship between (1.3) and (1.4). We also relate the corresponding q-traces (1.5) and (1.6) and the corresponding n-point functions (1.7) and (1.8).

(ii) In Section 3 we apply our results from Section 2 in the case of the infinite-wedge vertex operator (super)algebra $\mathscr{F}$ and its charge m subspace $\mathscr{F}_m$. It is well-known that these subspaces are $\hat{\mathscr{D}}$-modules. We compute q-traces (1.5) and (1.6) in the most interesting case (when the vectors u_i and v_i are chosen to be the fermionic generators) and the corresponding n-point functions. In particular, the n-point correlation functions (1.8) rise to the n-point functions introduced by Bloch and Okounkov [3]. The rest of Section 3 is devoted to studies of the q-difference equations satisfied by these n-point functions. We obtain several explicit formulas for 1- and 2-point functions by using different methods.

(iii) In Section 4, along the lines of Section 3, we consider a fermionic vertex operator superalgebra and the corresponding n-point functions associated with it. This case rises to a representation of an *orthogonal* Lie algebra of differential operators on the circle $\hat{\mathscr{D}}^{-}$.

(iv) In Section 5, we consider a vertex operator algebra associated to a free boson and the corresponding n-point functions associated to the bosonic generators. This case was studied in [24] in connection with a *symplectic* Lie algebra of differential operators on the circle $\hat{\mathscr{D}}^+$ [2,24] and the zeta-regularization procedure.

(v) In Appendix A we prove certain elementary lemmas necessary to deal with q-difference equations in Sections 3–5. Finally, in Appendix B we gave a different proof of the so-called "recursion formula" for the q-graded traces originally due to Zhu [26].

2. Correlation functions

As in Part I [24] we will use formal calculus as developed in [10]. We denote by x, y, t, x_i, t_i and y_i, etc., commuting formal variables and we take the liberty of using the same notation when formal variables are replaced by complex numbers. From the context it should be clear whether the variables are formal or complex.

2.1. Normal ordering procedure

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra (see [10] for the definition) and $u, v \in V$. From the Jacobi identity for vertex operator algebras it follows that

$$Y(u(-1)v,x) = \overset{\bullet}{\bullet} Y(u,x)Y(v,x) \overset{\bullet}{\bullet} := Y^-(u,x)Y(v,x) + Y(v,x)Y^+(u,x), \tag{2.1}$$

where $Y^+(v,x) = \sum_{n \geqslant 0} v(n)x^{-n-1}$ is the singular and $Y^-(v,x) = \sum_{n<0} v(n)x^{-n-1}$ is the regular part of $Y(v,x)$. Also, we define

$$\overset{\bullet}{\bullet} Y(u,x)Y(v,y) \overset{\bullet}{\bullet} = Y^-(u,x)Y(v,y) + Y(v,y)Y^+(u,x).$$

Let us recall the definition of the X-operator as in the first part [24]:

$$X(u,x) = Y(x^{L(0)}u,x),$$

where u is a homogeneous vector. We extend this definition by the linearity for every $u \in V$. The X-operator also admits a splitting into the regular and singular part, but then $X^-(u,x) \neq Y^-(x^{L(0)}u,x)$. Thus we have two different normal orderings for X operators

$$\overset{\bullet}{\bullet}{}' X(u,x)X(v,x) \overset{\bullet}{\bullet}{}' = X^-(u,x)X(v,x) + X(v,x)X^+(u,x)$$

and

$$\overset{\bullet}{\bullet} X(u,x)X(v,x) \overset{\bullet}{\bullet} = Y^-(x^{L(0)}u,x)X(v,x) + X(v,x)Y^+(x^{L(0)}u,x).$$

It is more convenient to work with $\overset{\bullet}{\bullet}\,\overset{\bullet}{\bullet}$ instead of $\overset{\bullet}{\bullet}{}'\overset{\bullet}{\bullet}{}'$. If we suppose that

$$u, v \in V_1 \quad \text{and} \quad [v(0), u(n)] = 0,$$

for every n, then

$$\begin{aligned}
{}^{\bullet}_{\bullet}{}' X(u,x)X(v,x) {}^{\bullet}_{\bullet}{}' &= \left\{ \sum_{n \leqslant 0} u(n)x^{-n} \right\} X(v,x) + X(v,x) \left\{ \sum_{n>0} u(n)x^{-n} \right\} \\
&= \left\{ \sum_{n<0} u(n)x^{-n} \right\} X(v,x) + u(0)X(v,x) + X(v,x) \left\{ \sum_{n>0} u(n)x^{-n} \right\} \\
&= \left\{ x \sum_{n<0} u(n)x^{-n-1} \right\} X(v,x) + X(v,x)x \left\{ \sum_{n \geqslant 0} u(n)x^{-n-1} \right\} \\
&= xY^{-}(u,x)X(v,x) + X(v,x)xY^{+}(u,x) = {}^{\bullet}_{\bullet} X(u,x)X(v,x) {}^{\bullet}_{\bullet}, \qquad (2.2)
\end{aligned}$$

meaning that there is no ambiguity which normal ordering we are using. This case arises when we study bosons and fermions. [2]

Proposition 2.1. *Suppose that*

$$u(n)v = c_{u,v}\delta_{n,\mathrm{wt}(u)+\mathrm{wt}(v)-1}\mathbf{1}, \qquad (2.3)$$

where $c_{u,v} \in \mathbb{C}$. *Then*
(a)

$${}^{\bullet}_{\bullet} X(u,x_1)X(v,x_2) {}^{\bullet}_{\bullet} = {}^{\bullet}_{\bullet} X(v,x_2)X(u,x_1) {}^{\bullet}_{\bullet}$$

(b) *and*

$${}^{\bullet}_{\bullet}{}' X(u,x_1)X(v,x_2) {}^{\bullet}_{\bullet}{}' = {}^{\bullet}_{\bullet} X(u,x_1)X(v,x_2) {}^{\bullet}_{\bullet}.$$

We will not use this result in the rest of the paper so we leave the proof to the reader.

2.2. Iterated 2n-point functions

In what follows we shall always use the following binomial expansion conventions. An expression $1/(x-y)^k$, where x and y are formal variables, is understood to be expanded (by the binomial theorem) in non-negative powers of y. Note that the order of variables is important. Also we allow x or y (but not both) to be a complex number.

As in [9] we will be considering formal series of the form

$$\frac{g(x_1,\ldots,x_n)}{\prod_{i=1}^{n} x_i^{r_i} \prod_{j<k}(x_j-x_k)^{s_{jk}}} \in \mathbb{C}[[x_1^{\pm 1},\ldots,x_n^{\pm 1}]], \qquad (2.4)$$

where $g(x_1,\ldots,x_n) \in \mathbb{C}[x_1,\ldots,x_n]$.

[2] All results about normal ordering hold with minor modifications for the vertex operator superalgebras. In particular, the formula (2.1) has to be replaced by ${}^{\bullet}_{\bullet} Y(u,x)Y(v,x) {}^{\bullet}_{\bullet} := Y^{-}(u,x)Y(v,x) + (-1)^{p(u)p(v)}Y(v,x)Y^{+}(u,x)$.

It is well-known (cf. [9,10]) that

$$\langle u'_{n+1}, Y(u_1,x_1)\cdots Y(u_n,x_n)u_{n+1}\rangle = f(x_1,\ldots,x_n),$$

for some $f(x_1,\ldots,x_n)$ of the form (2.4). After we replace the formal variables with complex variables (2.4) converges to a rational function inside the domain $|x_1| > \cdots > |x_n| > 0$.

Let $p_i, s_j, p_{i,j}, r_{i,j}, s_{i,j}$ and $u_{i,j}$ be natural numbers and $g(x_i,t_j)_{i,j=1}^n \in \mathbb{C}[x_i,t_j]_{i,j=1}^n$. Then

$$f(x_i,t_j)_{i,j=1}^n = \frac{g(x_i,t_j)_{i,j=1}^n}{\prod_{i=1}^n (t_i-1)^{p_i} t_i^{r_i} \prod_{j=1}^n x_j^{s_j} \prod_{i<j} (t_i x_i - t_j x_j)^{p_{i,j}} (t_i x_i - x_j)^{r_{i,j}} (x_i - t_j x_j)^{s_{i,j}} (x_i - x_j)^{u_{i,j}}} \tag{2.5}$$

is a well-defined element of $\mathbb{C}[[x_1^{\pm 1},\ldots,x_n^{\pm 1},t_1^{\pm 1},\ldots,t_n^{\pm 1}]]$ where we expand the denominator by using binomial expansion. Viewed as a rational function (2.5) has poles on the divisors: $x_i = x_j$, $x_i = 0$, $t_i = 0$, $t_i = 1$, $x_i = t_j x_j / t_i$, $x_i = t_j x_j$ and $x_i = x_j t_i^{-1}$. It is also clear that

$$\langle u'_{n+1}, X(u_1,x_1t_1)X(v_1,x_1)\cdots X(u_n,x_nt_n)X(v_n,x_n)u_{n+1}\rangle = g(x_i,t_j)_{i,=1}^n, \tag{2.6}$$

where $f(x_1,\ldots,x_n,t_1,\ldots,t_n)$ is of the form (2.5). We may assume $t_{i_1} = \cdots = t_{i_n}$ for some $1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant n$; in this case (2.5) is still well-defined.

It is important to notice that in (2.5) we cannot perform the substitution $t_i = e^{y_i}$, $i = 1,\ldots,n$, since the binomial expansion convention is not applicable to the expansion $1/(e^y - 1)^k$. Rather we use a different convention. As before $1/(e^{y_i}x_i - e^{y_j}x_j)^k$ are expanded by using binomial expansion, and factors of the form

$$\frac{1}{(e^x-1)^k},$$

$k \in \mathbb{N}$ stand for the formal multiplicative inverse of $(e^x-1)^k$, i.e.

$$\frac{x^{-k}}{(1+(x/2!+x^2/3!+\cdots))^k} \in \mathbb{C}((x)),$$

where we use binomial expansion with respect to $x/2! + x^2/3! + \cdots$. Sometimes we will view $1/(e^y-1)^k$ as a meromorphic function in y. With these conventions we show

Proposition 2.2. *$f(x_i, e^{y_j})_{i=1,j=1}^n$ is a well-defined element of*

$$\mathbb{C}((y_1,\ldots,y_n))[[x_1^{\pm 1},\ldots,x_n^{\pm 1}]].$$

Proof. It is enough to show that in the expansion

$$f(x_i,e^{y_j})_{i=1,j=1}^n = \sum_\alpha x^\alpha f_\alpha(e^{y_i})_{i=1}^n,$$

(we use a multi index notation $\alpha=(\alpha_1,\ldots,\alpha_n)$) for each α, $f_\alpha(e^{y_i})_{i=1}^n$ is a well-defined element of $\mathbb{C}((y_1,\ldots,y_n))$.

$$
\begin{aligned}
&\mathrm{Coeff}_{x^\alpha}\frac{g(x_i,e^{y_j})_{i,j=1}^n}{\prod_{i<j}(e^{y_i}x_i-e^{y_j}x_j)^{p_{i,j}}(e^{y_i}x_i-x_j)^{r_{i,j}}(x_i-e^{y_j}x_j)^{s_{i,j}}(x_i-x_j)^{u_{i,j}}}\\
&\quad=\mathrm{Coeff}_{x_1^{\alpha_1}}\left(\mathrm{Coeff}_{x_2^{\alpha_2}}\left(\ldots\mathrm{Coeff}_{x_n^{\alpha_n}}\right.\right.\\
&\qquad\times\left.\left.\frac{g(x_i,e^{y_j})_{i,j=1}^n}{\prod_{i<j}(e^{y_i}x_i-e^{y_j}x_j)^{p_{i,j}}(e^{y_i}x_i-x_j)^{r_{i,j}}(x_i-e^{y_j}x_j)^{s_{i,j}}(x_i-x_j)^{u_{i,j}}}\right)\right)\\
&\quad=\mathrm{Coeff}_{x_1^{\alpha_1}}\left(\mathrm{Coeff}_{x_2^{\alpha_2}}\left(\ldots\mathrm{Coeff}_{x_{n-1}^{\alpha_{n-1}}}\right.\right.\\
&\qquad\times\left.\left.\frac{h(x_i,e^{y_j})_{i,j=1}^{n-1}}{\prod_{i<j}(e^{y_i}x_i-e^{y_j}x_j)^{p_{i,j}}(e^{y_i}x_i-x_j)^{r_{i,j}}(x_i-e^{y_j}x_j)^{s_{i,j}}(x_i-x_j)^{u_{i,j}}}\right)\right),
\end{aligned}
\tag{2.7}
$$

where h is some Laurent polynomial. By induction it follows that (2.7) is an element of $\mathbb{C}[[y_1,\ldots,y_n]]$. Thus $f_\alpha(e^{y_i})_{i=1}^n\in\mathbb{C}((y_1,\ldots,y_n))$. □

Definition 2.1. Let $u_i,v_j\in V$, for $i=1,\ldots,n+1$, $j=1,\ldots,n$ and $u'_{n+1}\in V'$. We define an iterated $2n$-point function as a formal matrix coefficient:

$$\langle u'_{n+1},X(Y[u_1,y_1]v_1,x_1)\cdots X(Y[u_n,y_n]v_n,x_n)u_{n+1}\rangle, \tag{2.8}$$

where

$$Y[u,y]=e^{\mathrm{wt}(u)y}Y(u,e^y-1). \tag{2.9}$$

Vertex operators of the form (2.9) have been introduced by Zhu in [26]. A quadruple $(V,Y[,],\mathbf{1},\tilde{\omega})$, where $\omega=\omega-(c/24)$, is a vertex operator algebra isomorphic to $(V,Y(,),\mathbf{1},\omega)$. Operators of the type $X(Y[u,y]v,x)$ were introduced in [21,22] in connection with the Riemann ζ-function (cf. [24]).

2.3. Associativity for iterated $2n$-point functions

The aim is to relate (2.8) and (2.6). Let us consider the simplest case when $n=1$. We know (cf. [24]) that

$$X(Y[u,y]v,x)={}^{\bullet}_{\bullet}X(u,e^yx)X(v,x){}^{\bullet}_{\bullet}+X(Y^+[u,y]v,x),$$

where

$$Y^+[u,y]=\sum_{n\geqslant 0}u(n)(e^y-1)^{-n-1}$$

and the normal ordering $\mathbf{:\,:}$ for X-operators is the one introduced in Section 2.1. Hence

$$\begin{aligned} X(Y[u,y]v,x_1) &= \mathbf{:}X(u,e^y x)X(v,x)\mathbf{:} + X(Y^+[u,y]v,x) \\ &= \mathbf{:}X(u,e^y x)X(v,x)\mathbf{:} + \sum_{i\geqslant 0} (e^y)^{wt(u)} \frac{X(u(i)v,x)}{(e^y-1)^{i+1}}. \end{aligned} \tag{2.10}$$

On the other hand

$$\begin{aligned} &X(u,tx)X(v,x) \\ &\quad = \mathbf{:}X(u,tx)X(v,x)\mathbf{:} + [Y^+((tx)^{L(0)}u,tx),X(v,x)] \end{aligned} \tag{2.11}$$

Now we work out the second term in (2.11).

$$\begin{aligned} &[Y^+((tx)^{L(0)}u,tx),X(v,x)] \\ &\quad = (tx)^{\mathrm{wt}(u)}x^{\mathrm{wt}(v)} \lim_{x_1\to x} \mathrm{Sing}_{x_1}[Y^+(u,tx_1),Y(v,x)] \\ &\quad = (tx)^{\mathrm{wt}(u)}x^{\mathrm{wt}(v)} \lim_{x_1\to x} \mathrm{Sing}_{x_1} \mathrm{Res}_{x_0}\, x^{-1}\delta\left(\frac{tx_1-x_0}{x}\right) Y(Y(u,x_0)v,x) \\ &\quad = (tx)^{\mathrm{wt}(u)}x^{\mathrm{wt}(v)} \lim_{x_1\to x} \mathrm{Sing}_{x_1} \sum_{i\geqslant 0} x^{-1}e^{-x_0\frac{\partial}{\partial(tx_1)}}\delta\left(\frac{tx_1}{x}\right) Y(u(i)v,x) \\ &\quad = (tx)^{\mathrm{wt}(u)}x^{\mathrm{wt}(v)} \lim_{x_1\to x} \sum_{i\geqslant 0} \frac{1}{(tx_1-x)^{i+1}}\, Y(u(i)v,x) \\ &\quad = t^{\mathrm{wt}(u)}x^{\mathrm{wt}(u)+\mathrm{wt}(v)} \sum_{i\geqslant 0} \frac{x^{-i-1}}{(t-1)^{i+1}}\, Y(u(i)v,x) \\ &\quad = \sum_{i\geqslant 0} t^{\mathrm{wt}(u)}x^{\mathrm{wt}(u)+\mathrm{wt}(v)-i-1}\frac{1}{(t-1)^{i+1}}\, Y(u(i)v,x) \\ &\quad = \sum_{i\geqslant 0} \frac{t^{\mathrm{wt}(u)}}{(t-1)^{i+1}} X(u(i)v,x), \end{aligned} \tag{2.12}$$

where Sing_{x_1} stands for the regular part with respect to x_1 and $\lim_{x_1\to x}$ stands for the formal substitution $x_1\mapsto x$ (cf. [10]).

It is obvious also that

$$\begin{aligned} &\langle u_2', \mathbf{:}X(u,tx)X(v,x)\mathbf{:}u_2\rangle \in \mathbb{C}[t,x,t^{-1},x^{-1}], \\ &\langle u_2', \mathbf{:}X(u,e^y x)X(v,x)\mathbf{:}u_2\rangle \in \mathbb{C}[e^y,x,e^{-y},x^{-1}], \end{aligned} \tag{2.13}$$

for $u_2\in M$ and $u_2'\in M'$.

If we combine (2.10)–(2.12) we obtain

Proposition 2.3. *There exist a polynomial g such that*

$$\langle u_2', X(u_1, tx)X(v_1, x)u_2\rangle = \frac{g(tx,t)}{t^r x^s (t-1)^p},$$

for some $p, r, s \in \mathbb{N}$, $g(x,y) \in \mathbb{C}[x,y]$. *Moreover,*

$$\langle u_2', X(Y[u_1, y]v_1, x)u_2\rangle = \frac{g(e^y x, x)}{(e^y)^r x^s (e^y-1)^p}.$$

In particular, when x and y are specialized to be complex variables

$$\langle u_2', X(Y[u_1, y]v_1, x)u_2\rangle$$

converges to a holomorphic function inside $|x| > 0$, $0 < |y| < 2\pi$.

Proof. We may assume that all vectors are homogeneous. Relations (2.13), (2.11) and (2.12) imply that

$$\langle u_2', X(Y[u_1, y]v_1, x)u_2\rangle = \frac{g(e^y x, x)}{(e^y)^r x^s (e^y-1)^p}.$$

The second statement follows from the above discussion and the fact that $1/(e^y-1)$, as a complex function, has a Laurent expansion $1/y + \cdots$ which converges inside $0 < |y| < 2\pi$. This expansion coincides with the expansion

$$\frac{1}{e^y-1} = \frac{1}{y(1+(e^y-1-y)/y)} = \frac{1}{y} - \frac{e^y-1-y}{y} + \cdots$$

exhibited according to our conventions. □

Let us consider the general case ($n \geqslant 2$). For every $n \in \mathbb{N}$ we define domains

$$\Omega_{n,n} = \{(x_1,\ldots,x_n,t_1,\ldots,t_n) \in \mathbb{C}^{2n}:$$

$$\times |t_1 x_1| > |x_1| > |t_2 x_2| > |x_2| > \cdots > |t_{n-1}x_{n-1}| > |x_n| > 0\}. \tag{2.14}$$

It is easy to see that (2.5) is analytic inside $\Omega_{n,n}$. Note that $|t_i| > 1$, holds for every $i = 1,\ldots,n$ and $(x,t) := (x_1,\ldots,x_n,t_1,\ldots,t_n) \in \Omega_{n,n}$. We have the following theorem (that we call *associativity*)

Theorem 2.1. *There exist a formal series* $f(x_i,t_j)_{i,j=1}^n$ *of the form* (2.5) *such that*

$$\langle u_{n+1}', X(u_1, x_1t_1)X(v_1, x_1)\cdots X(u_n, x_nt_n)X(v_n, x_n)u_{n+1}\rangle = f(x_i,t_j)_{i,j=1}^n \tag{2.15}$$

and

$$\langle u_{n+1}' X(Y[u_1, y_1]v_1, x_1)\cdots X(Y[u_n, y_n]v_n, x_n)u_{n+1}\rangle = f(x_i, e^{y_j})_{i,j=1}^n, \tag{2.16}$$

where $f(x_i, e^{y_j})_{i,j=1}^n$ *is a series viewed according to our expansion conventions. In particular, as complex functions,* (2.15) *and* (2.16) *have the same meromorphic continuation onto* $\mathbb{C}^{2n}$ *if we replace* e^{y_i} *by* t_i.

Proof. We know that

$$\langle u_{n+1}', X(u_1, x_1t_1)X(v_1, x_1)\cdots X(u_n, x_nt_n)X(v_n, x_n)u_{n+1}\rangle$$

$$
\begin{aligned}
&= \langle u'_{n+1}, ({}^{\bullet}_{\bullet}X(u_1,x_1t_1)X(v_1,x_1){}^{\bullet}_{\bullet} + F_1(t_1,x_1))\cdots \\
&\quad \cdots({}^{\bullet}_{\bullet}X(u_1,x_1t_1)X(v_1,x_1){}^{\bullet}_{\bullet} + F_n(t_n,x_n))u_{n+1}\rangle,
\end{aligned}
\tag{2.17}
$$

where

$$F_i(t_i,x_n) = [Y^+((t_ix_i)^{L(0)}u,tx),X(v,x_i)].$$

On the other hand

$$
\begin{aligned}
&\langle u'_{n+1}, X(Y[u_1,y_1]v_1,x_1)\cdots X(Y[u_n,y_n]v_n,x_n)u_{n+1}\rangle \\
&\quad = \langle u'_{n+1}, ({}^{\bullet}_{\bullet}X(u_1,e^{y_1}x_1)X(v_1,x_1){}^{\bullet}_{\bullet} + f_1(e^{y_1},x_1))\ldots \\
&\qquad \cdots({}^{\bullet}_{\bullet}X(u_n,e^{y_n}x_n)X(v_n,x_n){}^{\bullet}_{\bullet} + f_n(e^{y_n},x_n))u_{n+1}\rangle,
\end{aligned}
\tag{2.18}
$$

where

$$f_i(e^{y_i},x_i) = X(Y^+[u_i,y_i]v_i,x_i),$$

and the (operator valued) series on the right-hand side are obtained according to our expansion convention.

Because (2.15) is a sum of several terms of the form

$$
\begin{aligned}
&\langle u'_{n+1}, {}^{\bullet}_{\bullet}X(u_1,x_1t_1)X(v_1,x_1){}^{\bullet}_{\bullet}\cdots\frac{t_j^{\mathrm{wt}(w)}X(w,x_j)}{(t_j-1)^m} \\
&\quad \cdots{}^{\bullet}_{\bullet}X(u_n,x_nt_n)X(v_n,x_n){}^{\bullet}_{\bullet}u_{n+1}\rangle,
\end{aligned}
\tag{2.19}
$$

for some $m \geqslant 0$, the only thing we have to prove is that

$$\langle u'_{n+1}, {}^{\bullet}_{\bullet}X(u_1,x_1t_1)X(v_1,x_1){}^{\bullet}_{\bullet}\cdots X(w,x_j)\cdots{}^{\bullet}_{\bullet}X(u_n,x_nt_n)X(v_n,x_n){}^{\bullet}_{\bullet}u_{n+1}\rangle, \tag{2.20}$$

is an expansion of a meromorphic function with (possible) poles at $x_i=0$, $t_i=0$, $x_i=x_j$, $t_ix_i=x_j$, $i \neq j$ and $t_ix_i=t_jx_j$. If this is the case then (2.20) does not contribute with additional poles at $t_i=1$ so we are allowed to make a substitution $t_i=e^{y_i}$ in (2.20) and use the same expansion conventions (which are applicable!) and obtain

$$\langle u'_{n+1}, {}^{\bullet}_{\bullet}X(u_1,x_1e^{y_1})X(v_1,x_1){}^{\bullet}_{\bullet}\cdots X(w,x_j)\cdots{}^{\bullet}_{\bullet}X(u_n,x_ne^{y_n})X(v_n,x_n){}^{\bullet}_{\bullet}u_{n+1}\rangle. \tag{2.21}$$

But we know that (2.18) is a sum of several terms like

$$
\begin{aligned}
&\langle u'_{n+1}, {}^{\bullet}_{\bullet}X(u_1,x_1e^{y_1})X(v_1,x_1){}^{\bullet}_{\bullet}\cdots\frac{e^{y_j\mathrm{wt}(w)}X(w,x_j)}{(e^{y_j}-1)^m} \\
&\quad \cdots{}^{\bullet}_{\bullet}X(u_n,x_ne^{y_n})X(v_n,x_n){}^{\bullet}_{\bullet}u_{n+1}\rangle,
\end{aligned}
\tag{2.22}
$$

so we have the proof. $\square$

Claim. *The expression* (2.19) *is an expansion of a rational function with no poles at* $t_i = 1$.

Proof. If we use

$$:X(u_i, x_i t_i)X(v_i, x_i): = X^-(u_i, x_i t_i)X(v_i, x_i) + X(v_i, x_i)X^+(u_i, x_i t_i),$$

for $i = 1$ we get

$$\begin{aligned}&\langle u'_{n+1}, :X(u_1, x_1 t_1)X(v_1, x_1): \cdots X(w, x_j) \cdots :X(u_n, x_n t_n)X(v_n, x_n): u_{n+1}\rangle \\ &\quad = \langle u'_{n+1}, X^+(u_1, x_1 t_1)X(v_1, x_1): \cdots X(w, x_j) \cdots :X(u_n, x_n t_n)X(v_n, x_n): u_{n+1}\rangle \\ &\qquad + \langle u'_{n+1}, X(v_1, x_1)X^-(u_1, x_1 t_1) \cdots :X(u_n, x_n t_n)X(v_n, x_n): u_{n+1}\rangle. \end{aligned} \tag{2.23}$$

After we apply the same formula, for every i, we obtain several summons. A generic term is

$$\langle u'_{n+1}, X(v_1, x_1)X^-(u_1, x_1 t_1) \cdots X(w_j, x_j) \cdots X^+(u_n, x_n t_n)X(v_n, x_n)u_{n+1}\rangle. \tag{2.24}$$

It is important to notice that $X^+(u_i, t_i x_i)$ is always to the right of $X(v_i, x_i)$ and $X^-(u_i, t_i x_i)$ is to the left of $X(v_i, x_i)$.

Now, by using the commutator formula we can move $X^+(u_i, t_i x_i)$ all the way to the right (such that it acts on w_{n+1}) and $X^-(u_i, x_i)$ all the way to the left. This procedure will produce poles only at wanted places, i.e., no new poles at $t_i = 1$. After we apply $X^+(u_i, t_i x_i)$ to w_{n+1} we obtain only finitely many terms. The same thing happens with $X^-(u_i, x_i)$. We can repeat the same procedure for remaining terms. This is a finite procedure and at the end we end up with a rational function with no poles at $t_i = 1$.

The proof follows. □

We define an *n-point function* to be a formal matrix coefficient:

$$\langle u'_{n+1}, \mathrm{o}(Y[u_1, y_1]v_1) \cdots \mathrm{o}(Y[u_n, y_n]v_n)u_{n+1}\rangle, \tag{2.25}$$

where $\mathrm{o}(a) = a(\mathrm{wt}(a) - 1)$, i.e., we extract the zeroth coefficient in (2.8) with respect to the expansion in terms of x_i's. In parallel with the previous construction we will consider:

$$\langle u'_{n+1}, \mathrm{o}(X(u_1, t_1 x_1)X(v_1, x_1)) \cdots \mathrm{o}(X(u_n, t_n x_n)X(v_n, x_n))u_{n+1}\rangle, \tag{2.26}$$

where

$$\mathrm{o}(X(u, tx)X(v, x)) := \mathrm{Res}_x\, x^{-1} X(u, tx)X(v, x).$$

For $n \geqslant 1$, let

$$\Omega_{n,0} = \{(t_1, \ldots, t_n) \in \mathbb{C}^n \colon |t_i| > 1, \ i = 1, \ldots, n\}.$$

Then we have

Corollary 2.1. *The expression*

$$f(t_1,\dots,t_n)$$
$$= \langle u'_{n+1}, \mathrm{o}(X(u_1,t_1x_1)X(v_1,x_1))\cdots \mathrm{o}(X(u_n,t_nx_n)X(v_n,x_n))u_{n+1}\rangle. \tag{2.27}$$

converges to a holomorphic function inside $\Omega_{n,0}$ and it has a meromorphic extension to $\mathbb{C}^n$ with poles at $t_i=1$ and $t_i=0$.

$$g(y_1,\dots,y_n) = \langle u'_{n+1}, \mathrm{o}(Y[u_1,y_1]v_1)\cdots \mathrm{o}(Y[u_n,y_n]v_n)u_{n+1}\rangle \tag{2.28}$$

converges inside $0<|y_i|<2\pi$. Moreover, as meromorphic functions,

$$g(y_1,\dots,y_n) = f(t_1,\dots,t_n)|_{t_1=e^{y_1},\dots,t_n=e^{y_n}}, \tag{2.29}$$

Proof. From the Proposition 2.2 we know that (2.28) is well-defined. Here we make the following additional observation:

$$\mathrm{Coeff}_{x_1^0\cdots x_n^0}\left\{\frac{g(x_i,t_j)_{i,j=1}^n}{\prod_{i<j}(t_ix_i-x_j)^{r_{i,j}}(x_i-t_jx_j)^{s_{i,j}}(x_i-x_j)^{u_{i,j}}}\right\}$$

is a Laurent polynomial in t_i, $i=1,\dots,n$. Because of (2.6), it follows that (2.27) converges inside $|t_i|>1$. Similarly,

$$\mathrm{Coeff}_{x_1^0\cdots x_n^0}\left\{\frac{g(x_i,e^{y_j})_{i,j=1}^n}{\prod_{i<j}(e^{y_i}x_i-x_j)^{r_{i,j}}(x_i-e^{y_j}x_j)^{s_{i,j}}(x_i-x_j)^{u_{i,j}}}\right\}$$

is (the same) Laurent polynomial in $t_i=e^{y_i}$, $i=1,\dots n$ and it converges inside $0<|y_i|<2\pi$. Now, Theorem 2.1 implies the last statement in the corollary. □

Remark 2.1. In Proposition 2.3 we did not use the associativity (as stated in [9]). One might think that there is an easier proof (cf. [20,21]):

$$\begin{aligned} X(u,tx)X(v,x) &= (tx)^{\mathrm{wt}(u)}x^{\mathrm{wt}(v)}Y(u,tx)Y(v,x)\\ &\sim (t_1x)^{\mathrm{wt}(u)}x^{\mathrm{wt}(v)}Y(Y(u,tx-x)v,x)\\ &\sim (t_1x)^{\mathrm{wt}(u)}X(x^{-L(0)}Y(u,tx-x)x^{L(0)}v,x)\\ &\sim X(t^{L(0)}Y(u,t-1)v,x). \end{aligned} \tag{2.30}$$

Even though (2.30) implies the "right" answer (after we substitute e^y for t), the proof is not rigorous! Note that the term $X(t^{L(0)}Y(u,t-1)v,x)$ is a non-rigorous expression (this can be seen if we extract zeroth coefficient of t) compared with the left-hand side which is well-defined. However, if we use matrix coefficients this construction can be made completely rigorous.

2.4. q-traces for the iterated 2n-point functions

In the previous section we showed that two correlation functions are related by a simple substitution (Theorem 2.1). Here we extend the same result for the corresponding q-traces. Let us consider (cf. [26]) a formal series (q-trace):

$$\mathrm{tr}|_M X(a_1,x_1)\cdots X(a_n,x_n)q^{L(0)} \in q^h\mathbb{C}[[x_1^{\pm 1},\ldots,x_n^{\pm 1},q]], \tag{2.31}$$

$a_i \in V$, $i=1,\ldots,n$, where M is a V-module which admits $\mathbb{N}$-grading, i.e., there is $h\in\mathbb{C}$ such that $\mathrm{Spec}(L(0)) \in h+\mathbb{Z}_{\geqslant 0}$. For $u_i, v_i \in V$, $i=1,\ldots,n$ we define a q-trace for an iterated $2n$-point function as a formal expression

$$\mathrm{tr}|_M X(Y[u_1,y_1]v_1,x_1)\cdots X(Y[u_n,y_n]v_n,x_n)q^{L(0)}. \tag{2.32}$$

Later $q=e^{2\pi i\tau}$, where $\tau\in\mathbb{H}$ ($\mathbb{H}$ is the upper half-plane).

It is a novelty, comparing to (2.31), to consider correlation functions of the form (2.32). Now the vector a_i is replaced by the formal expression $Y[u_i,y]v_i$. Thus one can think of (2.32) as a q-trace attached to a certain completion of V.

We will work intensively with another related q-trace:

$$\mathrm{tr}|_M X(u_1,t_1x_1)X(v_1,x_1)\cdots X(u_n,t_nx_n)X(u_n,x_n)q^{L(0)}. \tag{2.33}$$

By the definition

$$\begin{aligned}&\mathrm{tr}|_M X(u_1,t_1x_1)X(v_1,x_1)\cdots X(u_n,t_nx_n)X(v_n,x_n)q^{L(0)}\\ &\quad=\sum_{i\in I}\langle w_i',X(u_1,t_1x_1)X(v_1,x_1)\cdots X(u_n,t_nx_n)X(v_n,x_n)w_i\rangle q^{\mathrm{wt}(w_i)},\end{aligned} \tag{2.34}$$

where $\{w_i\}_{i\in I}$ is some (or any) homogeneous basis for M such that $\langle w_i',w_i\rangle=1$. We know that

$$\langle w_i',X(u_1,t_1x_1)X(v_1,x_1)\cdots X(u_n,t_nx_n)X(v_n,x_n)w_i\rangle$$

converges to an analytic function in the domain

$$|t_1x_1|>|x_1|>\cdots>|t_nx_n|>|x_n|>0,$$

but a priori we do not know whether (2.33) has a domain of convergence. To prove the convergence it suffices (according to [26]) to check that for every $v\in V$ the formal 1-point function

$$\mathrm{tr}|_M X(v,x)q^{L(0)}$$

converges to a holomorphic function inside $|q|<1$. This statement is very strong and it depends on the internal structure of the vertex operator algebra (C_2-condition for instance). If this is the case then one can show that (2.34) converges to a holomorphic function inside

$$1>|t_1x_1|>|x_1|>\cdots>|t_nx_n|>|x_n|>|q|, \tag{2.35}$$

and it has a meromorphic extension to $(\mathbb{C}^\times)^{2n}$.

In our approach 1-point function does not play a prominent role (actually we start with 2-point functions). Also we do not assume rationality, C_2-condition, etc. Therefore we need to develop a right recursion procedure for expressing q-traces of iterated $2n$-point functions in terms of q-traces of certain $(2n-2)$-point functions. Then, by using this result we prove the convergence of the iterated $2n$-point. We shall adopt this approach at the very end.

From now on we will assume that (2.3) holds for all pairs of vectors $u_i, v_j \in V$, $i = 1, \dots, n$. The following fact follows immediately from Theorem 2.1.

Corollary 2.2. *Let us denote* (2.33) *by* $f(t,x,q)$ *and* (2.32) *by* $g(y,x,q)$. *Then*

$$g(y,x,q) = f(e^y, x, q). \tag{2.36}$$

If we assume that x, y *and* t *are complex variables and* $q = e^{2\pi i \tau}$ *then* (2.36) *holds, provided that both sides are absolutely convergent in a certain domain.*

The previous corollary is not so useful since we do not have any information about poles.

In the case $n = 1$ we have the following description [26] (for the notation and a different proof see Appendix A).

Theorem 2.2. *Formally,*

$$\begin{aligned} &\mathrm{tr}|_M X(Y[u,y]v,x) \\ &\quad = \sum_{m \geqslant 1} \bar{\wp}_{m+1}(y,q)\, \mathrm{tr}|_M X(u[m]v,x) q^{L(0)} + \mathrm{tr}|_M \mathrm{o}(u)\mathrm{o}(v) q^{L(0)}, \end{aligned} \tag{2.37}$$

where $\bar{\wp}_{m+1}(y)$ *(defined in the Appendix* B*) are considered as elements from* $\mathbb{C}((y))[[q]]$.

Remark 2.2. All results obtained in the previous section carry out (in a straightforward manner) for vertex operator superalgebras (cf. [16]). Because of the $\frac{1}{2}\mathbb{Z}$-grading, correlation functions are multivalued functions and therefore the statements in Theorem 2.1 and Corollary 2.2 are ambiguous. If we assume that $p(u_i) = p(v_i)$ (where ε is the sign), in (2.33) possible multivaluedness stems only from the term

$$\sqrt{t_{i_1} \dots t_{i_k}},$$

for some $1 \leqslant i_i < \cdots < i_k \leqslant n$.

3. q-traces and $\hat{\mathscr{D}}$; charged fermions

3.1. Charged free fermions

In this section we exploit our general consideration in the case of the vertex operator superalgebra stemming from the pair of charged fermions [12,13]. Also, we discuss an equivalent bosonic construction.

First we introduce free (charged) fermions. We consider a Lie superalgebra generated by ψ_n and ψ^*_m, $n,m \in \mathbb{Z}+1/2$, and 1 with commutation relations

$$[\psi_n, \psi^*_m] = \delta_{m+n,0}, \quad [\psi^*_n, \psi^*_m] = [\psi_n, \psi_m] = 0.$$

We denote the corresponding Fock space by $\mathscr{F}$ [12] spanned by elements

$$\psi^*(-n_k)\cdots\psi^*(-n_1)\psi(-m_l)\cdots\psi(-m_1)\mathbf{1}, \tag{3.1}$$

where $n_i, m_j \in \mathbb{N}+\frac{1}{2}$ and $n_k > \cdots > n_1 \geqslant 1/2$, $m_l > \cdots > m_1 \geqslant \frac{1}{2}$.

Let us define the following vertex operators (fermionic fields):

$$Y(\psi(-1/2)\mathbf{1}, x) = \sum_{n\in\mathbb{Z}} \psi(n+1/2)x^{-n-1}$$

and

$$Y(\psi^*(-1/2)\mathbf{1}, x) = \sum_{n\in\mathbb{Z}} \psi^*(n+1/2)x^{-n-1}.$$

We write shorthand $\psi = \psi(-1/2)\mathbf{1}$ and $\psi^* = \psi^*(-1/2)\mathbf{1}$. Since the fermionic fields are local and generate the space $\mathscr{F}$ we can equip $\mathscr{F}$ with a structure of vertex operator superalgebra (see [8]) with the conformal vector

$$\omega = \frac{1}{2}\,\psi^*(-3/2)\psi(-1/2)\mathbf{1} + \frac{1}{2}\,\psi(-3/2)\psi^*(-1/2)\mathbf{1}.$$

This vertex operator superalgebra has the *charge* decomposition

$$\mathscr{F} = \bigoplus_{n\in\mathbb{Z}} \mathscr{F}_n,$$

with respect to the operator

$$\mathrm{o}(\,{\bullet\atop\bullet} X(\psi, x)X(\psi^*, x) {\bullet\atop\bullet}\,).$$

Then $\mathscr{F}_0$ is a vertex operator algebra and every $\mathscr{F}_n$ is an $\mathscr{F}_0$-module.

The projective representation of the Lie algebra $\mathscr{D}$ of differential operators on $\mathbb{C}^\times$ (see [8,14]), studied in [3], can be interpreted in the language of the vertex operator algebras in the following way.

Consider the following vectors:

$$\psi(-i-1/2)\psi^*(-j-1/2)\mathbf{1}, \tag{3.2}$$

$i, j \in \mathbb{N}$. Then $\mathscr{D}$ has a projective representation in terms of the Fourier coefficients of the vectors (3.2) (see [8]). Let $\mathscr{D}_0$ be the Cartan subalgebra of $\mathscr{D}$ (see [14]) spanned by $L_0^{(r)} = (t\mathrm{d}/\mathrm{d}t)^r$, $r \geqslant 0$.

We define an action of $\mathscr{D}_0$ in terms of generating functions (we choose a particular lifting) in the following way:

$$L_0^{(r)} \mapsto \bar{L}^{(r)}(0),$$

where $\bar{L}^{(r)}(0)$ is the coefficient of $x^0 y^r$ in

$$:X(\psi, e^y x)X(\psi^*, x): + \frac{e^{y/2}}{e^y - 1}. \tag{3.3}$$

According to [2,21,24], (3.3) corresponds to a new normal ordering. The operator $\bar{L}^{(r)}(0)$ introduced in [3] coincide with $\tilde{D}_r$ for every $r \in \mathbb{N}$. It is easy to see that operators $\bar{L}^{(r)}(0)$ act semisimply on $\mathscr{F}$ and therefore we can define a generalized character of $\mathscr{F}$ (cf. [8]):

$$\mathrm{tr}|_{\mathscr{F}} \prod_{i=0}^{\infty} q_i^{\bar{L}^{(i)}(0)}, \tag{3.4}$$

where $q_i = e^{2\pi i \tau_i}$, for $i \geqslant 1$ where $\tau_1 \in \mathbb{H}$, and q_0, τ_i, $i \geqslant 2$ are formal variables. Notice that $\bar{L}^{(0)}(0)$ is not associated to any particular element of $\hat{\mathscr{D}}$.

It is important to notice that (3.3) can be written as

$$X(Y[\psi, y]\psi^*, x). \tag{3.5}$$

Remark 3.1. One can derive a closed formula for the commutator

$$[X(Y[\psi, y_1]\psi, x_1), X(Y[\psi, y_2]\psi^*, x_2)], \tag{3.6}$$

by using the Jacobi identity from [24] or (3.3). We leave this (non-trivial) exercise to the reader.

Then

$$\mathrm{tr}|_{\mathscr{F}} \prod_{i=0}^{\infty} q_i^{\bar{L}^{(i)}(0)} = \prod_{i=1}^{\infty} q_{2i-1}^{\zeta(-2i+1,1/2)}$$

$$\times \prod_{n \geqslant 0} (1 + q_0 q_1^{n+1/2} q_2^{(n+1/2)^2} \cdots)(1 + q_0^{-1} q_1^{n+1/2} q_2^{-(n+1/2)^2} \cdots). \tag{3.7}$$

This corresponds to formula (1.12) in [3], where the same generating function is denoted by $\Omega(\tau_0, \tau_1, \ldots)$. The infinite product has an expansion as a Laurent series of q_0. We denote by $V(\tau_1, \tau_2 \ldots)$ the coefficient of q_0^0 of the generating function (3.7). Ω and V are *quasi-modular forms* (for the definition of quasi-modularity see [3,24]) of the weight 0 and $-\frac{1}{2}$, respectively.

3.2. Boson–fermion correspondence

Let L be a rank one lattice with a generator α such that $\langle \alpha, \alpha \rangle = 1$. We denote by $\hat{L}$ the non-trivial central extension of L

$$1 \to \{\pm 1\} \to \hat{L} \xrightarrow{-} L \to 0,$$

such that $ab = (-1)^{\langle \bar{a}, \bar{b} \rangle} ba$, $a, b \in \hat{L}$.

As in [6] (or [12]) we equip the space

$$V_L \cong M(1) \otimes \mathbb{C}[L], \tag{3.8}$$

with a structure of vertex operator superalgebra. Then

$$X(a^{\pm 1},x)=\exp\left(\sum_{n>0}\frac{\alpha(-n)}{-n}x^{n}\right)\exp\left(\sum_{n>0}\frac{\alpha(n)}{n}x^{-n}\right)a^{\pm 1}z^{\pm\bar{a}}, \tag{3.9}$$

where $\bar{a}=\alpha$. The boson–fermion correspondence (cf. [12]) is a vertex operator superalgebra isomorphism

$$\Psi:\mathscr{F}\to V_L, \tag{3.10}$$

with the following properties:

$$\psi(-n+1/2)\cdots\psi(-1/2)\mathbf{1}\mapsto e^{n\alpha}$$

$$\psi^*(-n+1/2)\cdots\psi^*(-1/2)\mathbf{1}\mapsto e^{-n\alpha},$$

$$\omega\mapsto\frac{1}{2}\alpha(-1)^2\mathbf{1}. \tag{3.11}$$

In particular, $\psi(-1/2)\mathbf{1}\mapsto e^{\alpha}$ and $\psi^*(-1/2)\mathbf{1}\mapsto e^{-\alpha}$.

Now (3.5) can be expressed in the following form:

$$X(Y[\psi,y]\psi^*,x)=\Psi^{-1}X(Y[e^{\alpha},y]e^{-\alpha},x)\Psi, \tag{3.12}$$

since

$$\Psi(Y[\psi,y]\psi^*)=Y[e^{\alpha},y]e^{-\alpha},$$

which is a consequence of Proposition 4.1 in [24].

If we denote by $\tilde{L}^{(r)}(0)$ the coefficient of x^0y^r in

$$X(Y[a,y]a^{-1},x),$$

then

$$\mathrm{tr}|_{\mathscr{F}}\prod_{i=0}^{\infty}q_i^{\bar{L}^{(i)}(0)}=\mathrm{tr}|_{V_L}\prod_{i=0}^{\infty}q_i^{\tilde{L}^{(i)}(0)}. \tag{3.13}$$

In particular if we consider only the coefficient of q_0^0, i.e. $V(\tau_1,\tau_2,\ldots)$, we get

$$\mathrm{tr}|_{\mathscr{F}_0}\prod_{i=0}^{\infty}q_i^{\bar{L}^{(i)}(0)}=\mathrm{tr}|_{M(1)}\prod_{i=0}^{\infty}q_i^{\tilde{L}^{(i)}(0)}, \tag{3.14}$$

where $M(1)$ is identified with $M(1)\otimes 1\subset V_L$ by means of (3.8).

Remark 3.2. Generalized characters discussed in this section already appeared in the literature. In [4,5] similar generating functions related to mirror symmetry on the torus resemble (3.7). Also they appear (with finitely many $\bar{L}^{(r)}(0)$'s) in [18].

3.3. *Bloch–Okounkov's n-point function*

In [3] Bloch and Okounkov introduced another type of generating functions. Let us associate to the generalized character $V(\tau_1,\tau_2,\ldots)$ an *n-point function* in the following way. First we define operators

$$\sigma(y)=\frac{1}{y}+\frac{1}{2\pi i}\sum_{r=1}^{\infty}\frac{\partial}{\partial\tau_r}\frac{y^r}{r!}.$$

Then the (formal) n-point function is defined as

$$\mathscr{F}(y_1,\ldots,y_n)=\eta(q_1)\sigma(y_1)\cdots\sigma(y_n)V(\tau_1,\tau_2,\ldots)|_{\tau_2=\tau_3=\cdots 0}. \tag{3.15}$$

We mention that u_i's is a formal variables in contrast to [3] where u_i is a complex variable.

Proposition 3.1. *We have*

$$\begin{aligned}
&\eta(q_1)\sigma(y_1)\cdots\sigma(y_n)V(\tau_1,\tau_2,\ldots)|_{\tau_2=\tau_3=\cdots 0}\\
&\quad=\mathrm{Coeff}_{x^0}\,\mathrm{tr}|_{\mathscr{F}_0}\eta(q_1)X(Y[\psi,y_1]\psi^*,x_1)\cdots X(Y[\psi,y_n]\psi^*,x_n)q_1^{\bar{L}(0)}\\
&\quad=\mathrm{tr}|_{\mathscr{F}_0}\eta(q_1)\mathrm{o}(Y[\psi,y_1]\psi^*)\cdots\mathrm{o}(Y[\psi,y_n]\psi^*)q_1^{\bar{L}(0)},
\end{aligned} \tag{3.16}$$

where $\bar{L}(0)=L(0)-\frac{1}{24}$.

Proof. The second equality is trivial since

$$X(u,x)=\sum_{n\in\mathbb{Z}}u(\mathrm{wt}(u)-n-1)x^{-n}.$$

For the first equality we prove only in the case $n=1$. Let us recall

$$V(\tau_1,\tau_2,\ldots)=\mathrm{tr}|_{\mathscr{F}_0}\prod_{i=1}^{\infty}q_i^{\bar{L}^{(i)}(0)}.$$

Then

$$\begin{aligned}
&\eta(q_1)\sigma(y)V(\tau_1,\tau_2,\ldots)|_{\tau_2=\tau_3=\cdots 0}\\
&\quad=\mathrm{tr}|_{\mathscr{F}_0}\eta(q_1)\left(\frac{1}{y}+\sum_{r=1}^{\infty}\bar{L}^{(r)}(0)\frac{y^r}{r!}\right)q_1^{\bar{L}(0)}\\
&\quad=\mathrm{tr}|_{\mathscr{F}_0}\eta(q_1)\left(\frac{1}{y}-\sum_{r=1}^{\infty}\zeta(-r,1/2)\frac{y^r}{r!}+\mathrm{o}({}^{\bullet}_{\bullet}X(\psi,e^yx)X(\psi,x){}^{\bullet}_{\bullet})\right.\\
&\qquad\left.-\,\mathrm{o}({}^{\bullet}_{\bullet}X(\psi,x)X(\psi,x){}^{\bullet}_{\bullet})\right)q_1^{\bar{L}(0)}.
\end{aligned} \tag{3.17}$$

Since the charge operator $\mathrm{o}({}^{\bullet}_{\bullet}X(\psi,x)X(\psi,x){}^{\bullet}_{\bullet})$ acts as zero on $\mathscr{F}_0$ and

$$\frac{1}{y}-\sum_{r=1}^{\infty}\zeta\left(-r,\frac{1}{2}\right)\frac{y^r}{r!}=\frac{e^{y/2}}{e^y-1},$$

the proof follows from (3.5). □

Now let us apply the results from Section 2.4 in the case of vertex operator superalgebra $\mathscr{F}$ (with the obvious super-modification). Fix $u_i=\psi$, $v_i=\psi^*$, $i=1,\dots,n$. Notice that (unlike in Section 2.4.) we do not consider an $\mathscr{F}$-module but rather a subalgebra of $\mathscr{F}$. Since $X(\psi, t_i x_i)X(\psi^*,x_i)$ acts on $\mathscr{F}_0$ we can compute the q-trace

$$\mathrm{tr}|_{\mathscr{F}_0} X(\psi, t_1x_1)X(\psi^*,x_1)\cdots X(\psi, t_nx_n)X(\psi^*,x_n)q^{\bar{L}(0)}, \tag{3.18}$$

and a corresponding coefficient of x^0, multiplied by $\eta(q)$:

$$G(t_1,\dots,t_n) := \eta(q)\,\mathrm{tr}|_{\mathscr{F}_0} \mathrm{o}(X(\psi, t_1x_1)X(\psi^*,x_1))\cdots \mathrm{o}(X(\psi, t_nx_n)X(\psi^*,x_n))q^{\bar{L}(0)}. \tag{3.19}$$

Proposition 3.2. *Suppose that (3.19) converges, in some domain, to a multivalued function $G(t_1,\dots,t_n)$. Then, as complex functions,*

$$\mathscr{F}(y_1,\dots,y_n) = G(t_1,\dots,t_n)|_{t_i^{n/2}=e^{ny_i/2}}$$

for every y provided that the right-hand side is convergent.

Proof. Since

$$\psi(n)\psi^* = \delta_{n,1/2}\mathbf{1},$$

for $n \geqslant \frac{1}{2}$ and

$$X(\psi, t_ix_i)X(\psi^*,x_i) = {}^{\bullet}_{\bullet}X(\psi, t_ix_i)X(\psi^*,x_i){}^{\bullet}_{\bullet} + \frac{t^{1/2}}{t-1},$$

$$X(Y[\psi, y_i]\psi^*,x_i) = {}^{\bullet}_{\bullet}X(\psi, e^{y_i}x_i)X(\psi^*,x_i){}^{\bullet}_{\bullet} + \frac{e^{y_i/2}}{e^{y_i}-1},$$

we can apply Theorem 2.1 and Corollary 2.1 (with minor modifications). The correlation function G converges to a multi valued function (cf. Remark 2.2). However, we can choose a branch $t_i^{n/2} = e^{ny_i/2}$. Therefore the Proposition holds. $\square$

We will need the following trivial (but important) observation.

Lemma 3.1. *Let $\varphi: V_1 \to V_2$ be a vertex operator superalgebra isomorphism and W a subalgebra of V_1 stable under $X(u_1,x_1)\cdots X(u_n,x_n)$. Then*

$$\mathrm{tr}|_W X(u_1,x_1)\cdots X(u_n,x_n)q^{L_{V_1}(0)} = \mathrm{tr}|_{\varphi(W)} X(\varphi(u_1),x_1)\cdots X(\varphi(u_n),x_n)q^{L_{V_2}(0)}. \tag{3.20}$$

Proof. It follows immediately form the property of the trace

$$\mathrm{tr}|_{\varphi(W)}\varphi X(u_1,x_1)\cdots X(u_n,x_n)\varphi^{-1}q^{L_{V_2}(0)} = \mathrm{tr}|_M X(u_1,x_1)\cdots X(u_n,x_n)q^{L_{V_1}(0)},$$

and the fact that

$$\varphi X(u_1,x_1)\varphi^{-1} = X(\varphi(u_1),x_1). \qquad \square$$

3.4. q-traces in the case of free fermions

The boson–fermion correspondence implies (cf. Lemma 3.1) that

$$\mathrm{tr}|_{\mathscr{F}} X(\psi, t_1x_1)X(\psi^*, x_1)\cdots X(\psi, t_nx_n)X(\psi^*, x_n)q^{\bar{L}(0)}, \tag{3.21}$$

is equal to

$$\mathrm{tr}|_{V_L} X(a, t_1x_1)X(a^{-1}, x_1)\cdots X(a, t_nx_n)X(a^{-1}, x_n)q^{\bar{L}(0)}, \tag{3.22}$$

and the corresponding trace for the zero-charge subspace $\mathscr{F}_0$ is equal to (3.18).

Now we need some notation. Let $(q)_\infty = \prod_{i\geqslant 1}^{\infty}(1-q^i)$ and

$$\theta(t) = \sum_{n\in\mathbb{Z}} q^{(n^2/2)}t^n \in \mathbb{C}[[q^{1/2}, t^{\pm 1}]].$$

If $t = e^{2\pi i v} \in \mathbb{C}^\times$ and $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$ then $\theta(t)$ is the classical theta function. Also we define another theta function in the infinite-product form

$$\theta_{11}(\tau, v) = \frac{\prod_{i=1}^{\infty}(1-q^it)\prod_{i=1}^{\infty}(1-q^it^{-1})(t^{1/2}-t^{-1/2})}{(q)_\infty^2}.$$

This is an entire function of v for fixed $\tau \in \mathbb{H}$. For a fixed τ we will write $\theta_{11}(v)$. These theta functions can be thought as functions of t (instead of v) but then these are multi valued function which we will denote by $\theta(t)$ and $\theta_{11}(t)$. The following transformation formulas hold:

$$\theta(qt) = q^{-1/2}t^{-1}\theta(t),$$

$$\theta_{11}(qt) = -q^{-1/2}t^{-1}\theta(t).$$

The following result has been known, in some form, for a while. For the completeness we include a proof here.

Theorem 3.1.

(a)

$$\mathrm{tr}|_{\mathscr{F}} X(\psi, t_1x_1)X(\psi^*, x_1)\cdots X(\psi, t_nx_n)X(\psi^*, x_n)q^{L(0)} \tag{3.23}$$

converges to a multi-valued holomorphic function in the domain

$$|t_1x_1| > |x_1| > \cdots > |t_nx_n| > |x_n| > |qt_1x_1| > 0, \tag{3.24}$$

(b)

$$\eta(q)\,\mathrm{tr}|_{\mathscr{F}_0} X(\psi, t_1x_1)X(\psi^*, x_1)\cdots X(\psi, t_nx_n)X(\psi^*, x_n)q^{\bar{L}(0)} \tag{3.25}$$

converges in the domain (3.24), *and it has a meromorphic extension to a double cover of* $(\mathbb{C}^\times)^{2n}$ *equals to*

$$\frac{\prod_{i<j}\theta_{11}(t_jx_j/t_ix_i)\theta_{11}(x_j/x_i)}{\prod_{i<j}\theta_{11}(t_jx_j/x_i)\theta_{11}(x_j/t_ix_i)\prod_{i=1}^{n}\theta_{11}(t_i)}. \tag{3.26}$$

Proof. Let

$$G_n(t,x) := X(\psi, x_1 t_1)X(\psi^*, t_1)\cdots X(\psi, x_n t_n)X(\psi^*, t_n).$$

First notice that, because of the boson–fermion correspondence, we can replace $X(\psi, x_1 t_1)X(\psi^*, t_1)$ by $X(a, x_1 t_1)X(a^{-1}, t_1)$. Also

$$\mathrm{tr}|_{M(1)\otimes\mathbb{C}[L]} G_n(t,x)q^{L(0)} = (\mathrm{tr}|_{M(1)} G_n(t,x)q^{L(0)})(\mathrm{tr}_{\mathbb{C}[L]} G_n(t,x)q^{L(0)}). \tag{3.27}$$

We fix a basis for $M(1)$:

$$\frac{h(-n_1)^{k_1}\cdots h(-n_l)^{k_l}\mathbf{1}}{\sqrt{\prod_{i=1}^{l} k_i! n_i^{k_i}}},$$

where $h = \frac{\alpha}{\sqrt{2}}$, $n_1 > n_2 > \cdots > n_l$ and $k_1, \ldots, k_l \in \mathbb{N}$. Also we fix the corresponding dual basis

$$\frac{h'(-n_1)^{k_1}\cdots h'(-n_l)^{k_l}\mathbf{1}}{\sqrt{\prod_{i=1}^{l} k_i! n_i^{k_i}}}.$$

Now we calculate (as in [25])

$$\begin{aligned}
&\mathrm{tr}_{\mathbb{C}[h(-m)]} G_n(t,x)q^{L(0)} \\
&\quad = \sum_{k\in\mathbb{N}} \frac{1}{k! m^k} \langle h'(-m)^k, G_n(t,x)q^{L(0)} h(-m)^k\rangle \\
&\quad = \sum_{m\in\mathbb{N}} \frac{\prod_{i<j}((t_j x_j/t_i x_i)^{1/2} - (t_j x_j/t_i x_i)^{-1/2})((x_j/x_i)^{1/2} - (x_j/x_i)^{-1/2})}{\prod_{i=1}^{n}(t_i^{1/2} - t_i^{-1/2})\prod_{i<j}((t_j x_j/x_i)^{1/2} - (t_j x_j/x_i)^{-1/2})((x_j/t_i x_i)^{1/2} - (x_j/t_i x_i)^{-1/2})} \\
&\qquad \times \frac{1}{k! m^k} \langle h'(-m)^k {\bf :} G_n(t,x) {\bf :} h(-m)^k\rangle.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{k\in\mathbb{N}} \frac{1}{k! m^k} \langle h'(-m)^k {\bf :} G_n(t,x) {\bf :} h(-m)^k\rangle \\
&\quad = \sum_{k\in\mathbb{N}} \frac{1}{k! m^k} \langle h'(-m)^k, \exp\left(\frac{\alpha(-m)}{m}\sum_{r=1}^{n} (x_s t_s)^{-m} + x_s^{-m}\right) \\
&\quad = \exp\left(\frac{\alpha(m)}{-m}\sum_{i=1}^{n} (x_r t_r)^{nm} + x_r^{m}\right) h(-m)^k\rangle \\
&\quad = \sum_{k\in\mathbb{N}} \frac{1}{k! m^k} \sum_{i=0}^{k} \frac{1}{i!}\left(-\frac{1}{m}\sum_{r} (x_r t_r)^{-m} + x_r^{-m}\right)^i \langle h'(-m)^k, \exp\left(\frac{\alpha(-m)}{m}\right.
\end{aligned}$$

$$\times \left.\sum_{s=1}^{n}(x_s t_s)^{-m} + x_s^{-m}\right)\alpha(m)^i q^{mk} h(-m)^k\rangle$$

$$=\sum_{i=0}^{\infty}\sum_{k=i}^{\infty}\frac{k!\langle h'(-m)^k, h(-m)^k\rangle}{m^k k!}$$

$$\times\frac{(-1/m\sum_{r,s} t_r^m x_r^m/t_s^m x_s^m + x_r/x_s - t_r^m x_r^m/x_s - x_r/t_s^m x_s^m)^i}{(k-i)!i!^2}q^{mk}$$

$$=\sum_{i=0}^{\infty}\frac{(-1/m\sum_{r,s} t_r^m x_r^m/t_s^m x_s^m + x_r/x_s - t_r^m x_r^m/x_s - x_r/t_s^m x_s^m)^i q^{im}/(1-q^m)^i}{i!(1-q^m)}$$

$$=\exp\left(-\frac{1}{m}\sum_{r,s}\left(\frac{t_r^m x_r^m}{t_s^m x_s^m}+\frac{x_r}{x_s}-\frac{t_r^m x_r^m}{x_s}-\frac{x_r}{t_s^m x_s^m}\right)\frac{q^m}{1-q^m}\right)\frac{1}{1-q^m}. \tag{3.28}$$

So far we calculated the trace on the space $\mathbb{C}[h(-m)]$. Since

$$M(1)\cong\otimes_{m=1}^{\infty}\mathbb{C}[h(-m)],$$

then

$$\mathrm{tr}|_{M(1)}A=\prod_{m=1}^{\infty}\mathrm{tr}|_{\mathbb{C}[h(-m)]}A.$$

Therefore

$$\prod_{m=1}^{\infty}\exp\left(-\frac{1}{m}\left(\frac{t_r^m x_r^m}{t_s^m x_s^m}+\frac{t_s^m x_s^m}{t_r^m x_r^m}+1\right)\frac{q^m}{1-q^m}\right)$$

$$=\prod_{t=1}^{\infty}\frac{(1-q^t t_r x_r/t_s x_s)(1-q^t t_s x_s/t_r x_r)}{(1-q^t)^2}, \tag{3.29}$$

for $1\leqslant r\leqslant n$ and $1\leqslant s\leqslant n$. Thus

$$\frac{\prod_{i<j}((t_j x_j/t_i x_i)^{1/2}-(t_j x_j/t_i x_i)^{-1/2})((x_j/x_i)^{1/2}-(x_j/x_i)^{-1/2})}{\prod_{i=1}^{n}(t_i^{1/2}-t_i^{-1/2})\prod_{i<j}((t_j x_j/x_i)^{1/2}-(t_j x_j/x_i)^{-1/2})((x_j/t_i x_i)^{1/2}-(x_j/t_i x_i)^{-1/2})}$$

$$\times\prod_{m=1}^{\infty}\exp\left(-\frac{1}{m}\sum_{r,s}\left(\frac{t_r^m x_r^m}{t_s^m x_s^m}+\frac{x_r}{x_s}-\frac{t_r^m x_r^m}{x_s}-\frac{x_r}{t_s^m x_s^m}\right)\frac{q^m}{1-q^m}\right)\frac{1}{1-q^m}$$

$$=\frac{\prod_{i<j}\theta_{11}(t_j x_j/t_i x_i)\theta_{11}(x_j/x_i)}{\prod_{i<j}\theta_{11}(t_j x_j/x_i)\theta_{11}(x_j/t_i x_i)\prod_{i=1}^{n}\theta_{11}(t_i)(q)_\infty}. \tag{3.30}$$

If we assume that

$$|t_1x_1| > |x_1| > \cdots > |t_nx_n| > |x_n| > |qt_1x_1| > 0,$$

then the infinite product (3.29) converges uniformly to a holomorphic function inside (3.24). This is a proof of (b). For (a) we have to calculate (cf. (3.27))

$$\mathrm{tr}|_{\mathbb{C}[L]} G_n(t,x)q^{L(0)} = \sum_{n\in\mathbb{Z}} t_1^m \cdots t_n^m q^{m^2/2}.$$

which is convergent for $|q| < 1$. □

From Theorem 3.1 it follows that

$$\eta(q)\,\mathrm{tr}|_{\mathscr{F}_0} X(\psi, t_1x_1)X(\psi^*, x_1)\cdots X(\psi, t_nx_n)X(\psi^*, x_n)q^{L(0)}$$

has a meromorphic continuation (after we specify branches) to $(\mathbb{C}^\times)^{2n}$ with the set of singularities determined by zeros of

$$\prod_{i=1}^{n} \theta_{11}(t_i) \prod_{i<j} \theta_{11}(t_jx_j/x_i)\theta_{11}(x_j/t_ix_i). \tag{3.31}$$

Let $x = (x_1,\dots,x_n)$. Denote by $\Lambda_n(x)$ the set of all n-tuples $(t_1,\dots,t_n)$ such that $(t_1,\dots,t_n,x_1,\dots,x_n)$ satisfies inequality (3.24).

3.5. *Another description of the q-difference equations*

From Theorem 3.1(b) it follows:

Corollary 3.1. *Let* $|q| < 1$. *Then*

$$\mathrm{tr}|_{\mathscr{F}_0} \eta(q)X(a,tx)X(a^{-1},x)q^{L(0)} \tag{3.32}$$

converges in the domain $1/|q| > |t| > 1$. *Moreover*, (3.32) *has a meromorphic extension to a double cover of* $\mathbb{C}^\times$ *equal to*

$$\frac{1}{\theta_{11}(t)}.$$

Corollary 3.1 is essentially Theorem 6.5 in [3].

Remark 3.3. According to [3], the following formula holds:

$$\mathscr{F}(y_1,\dots,y_n) = \sum_{\sigma\in S_n} \frac{\det(\theta_{11}^{(i-j+1)}(t_{\sigma(1)}\cdots t_{\sigma(n-j)})/(j-i+1)!)_{i,j=1}^n}{\theta_{11}(t_{\sigma(1)})\dots\theta_{11}(t_{\sigma(1)}\dots t_{\sigma(n)})}, \tag{3.33}$$

where $t_i = e^{y_i}$. Let us denote by $G(t_1,\dots,t_n)$ the coefficient of x^0 in the expansion of (3.25). Because of Proposition 3.2

$$\mathscr{F}(y_1,\dots,y_n) = G(t_1,\dots,t_n)|_{t_i^{n/2}=e^{ny_i/2}}.$$

If we switch to complex variables a Laurent expansion of (3.26) (inside a certain domain) gives us an integral representation of $G(t_1,\ldots,t_n)$. Let A_i be an annulus inside the x_i-plane such that A_i is contained in (3.24), and $\mathscr{C}_i$ is a smooth curve inside A_i. Then the Cauchy's integral formula for several complex variables implies

$$G(t_1,\ldots,t_n) = \frac{1}{(2\pi i)^n} \oint_{C_1} \cdots \oint_{C_n} \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{x_1 \ldots x_n}$$
$$\times \frac{\prod_{i<j} \theta_{11}(t_j x_j/t_i x_i)\theta_{11}(x_j/x_i)}{\prod_{i<j} \theta_{11}(t_j x_j/x_i)\theta_{11}(x_j/t_i x_i) \prod_{i=1}^{n} \theta_{11}(t_i)}. \tag{3.34}$$

This integral is hard to calculate explicitly.

Let us consider the case $n=2$ in more details. Let $x=x_2/x_1$. Assume that $q^n = e^{2\pi n \tau}$, $\tau \in \mathbb{H}$ and $|t_2| > |t_1|$. Then the set of singularities (3.31) is ordered (with respect to the absolute value) in the following way:

$$\cdots > \left|\frac{q^k}{t_2}\right| > |t_1| > \left|\frac{q^{k+1}}{t_2}\right| > |qt_1| > \left|\frac{q^{k+2}}{t_2}\right| > \cdots, \tag{3.35}$$

for some $k \in \mathbb{Z}$. Suppose for simplicity that $k=0$, i.e.

$$\cdots > \left|\frac{1}{t_2}\right| > |t_1| > \left|\frac{q}{t_2}\right| > |qt_1| > \left|\frac{q^2}{t_2}\right| > \cdots. \tag{3.36}$$

Put $r_1 = |qt_1|$, $r_2 = |q/t_2|$ and $r_3 = |t_1|$. Let us denote by $\mathscr{C}_r(z) \in \mathbb{C}$ a circle with center at z and radius r. From Remark 3.3, in the case when $n=2$, it follows

Corollary 3.2. *Let*

$$|t_1| > \left|\frac{q}{t_2}\right|. \tag{3.37}$$

Then

$$G(t_1,t_2) = \frac{1}{\theta_{11}(t_1)\theta_{11}(t_2)} \oint_{\mathscr{C}_{a_1}(0)} \frac{\mathrm{d}x}{2\pi i x} \frac{\theta_{11}(t_2 x/t_1)\theta_{11}(x)}{\theta_{11}(t_2 x)\theta_{11}(x/t_1)}, \tag{3.38}$$

where

$$a_1 = \frac{|t_1| + |q|/|t_2|}{2}.$$

Let us try to determine the behavior of (3.38) under the elliptic transformation $t_1 \mapsto qt_1$. Now,

$$G(qt_1,t_2) = \frac{1}{\theta_{11}(qt_1)\theta_{11}(t_2)} \oint_{\mathscr{C}_{a_2}(0)} \frac{\mathrm{d}x}{2\pi i x} \frac{\theta_{11}(t_2 x/qt_1)\theta_{11}(x)}{\theta_{11}(t_2 x)\theta_{11}(x/qt_1)},$$

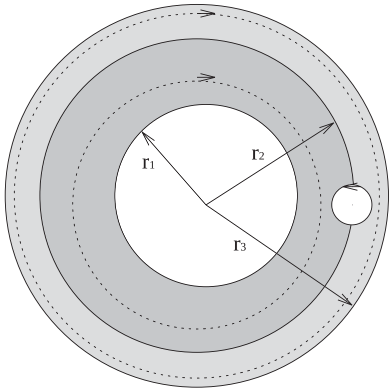

Fig. 1. Contours of integration.

where

$$a_2 = \frac{|q|/|t_2| + |qt_1|}{2}.$$

Since

$$G(qt_1,t_2,x) = -q^{1/2}t_1t_2G(t_1,t_2,x),$$
$$G(qt_1,t_2) = -\frac{q^{1/2}t_1t_2}{\theta_{11}(t_1)\theta_{11}(t_2)}\oint_{\mathscr{C}_{a_2}(0)}\frac{\mathrm{d}x}{2\pi ix}\,\frac{\theta_{11}(t_2x/t_1)\theta_{11}(x)}{\theta_{11}(t_2x)\theta_{11}(x/t_1)}.$$

Now from (3.36) and Cauchy's theorem we have

$$\begin{aligned} G(qt_1,t_2) &= -\frac{q^{1/2}t_1t_2}{\theta_{11}(t_1)\theta_{11}(t_2)}\oint_{\mathscr{C}_{a_2}(0)}\frac{\mathrm{d}x}{2\pi ix}\,\frac{\theta_{11}(t_2x/t_1)\theta_{11}(x)}{\theta_{11}(t_2x)\theta_{11}(x/t_1)} \\ &= -q^{1/2}t_1t_2\left(\frac{1}{\theta_{11}(t_1)\theta_{11}(t_2)}\oint_{\mathscr{C}_{a_1}(0)}\frac{\mathrm{d}x}{2\pi ix}\,\frac{\theta_{11}(t_2x/t_1)\theta_{11}(x)}{\theta_{11}(t_2x)\theta_{11}(x/t_1)}\right. \\ &\quad \left. + \frac{1}{\theta_{11}(t_1)\theta_{11}(t_2)}\oint_{\mathscr{C}_{\varepsilon}(q/t_2)}\frac{\mathrm{d}x}{2\pi ix}\,\frac{\theta_{11}(t_2x/t_1)\theta_{11}(x)}{\theta_{11}(t_2x)\theta_{11}(x/t_1)}\right), \end{aligned} \tag{3.39}$$

for some small $\varepsilon > 0$ (cf. Fig. 1, where dotted circles have radius a_1 and a_2 and the small circle is centered at q/t_2). Let us evaluate the second integral on the right-hand side of (3.39).

Lemma 3.2. *For small* $\varepsilon > 0$
(a)

$$\frac{1}{\theta_{11}(t_1)\theta_{11}(t_2)} \oint_{\mathscr{C}_\varepsilon(qt_1)} \frac{\mathrm{d}x}{2\pi i x} \frac{\theta_{11}(t_2 x/t_1)\theta_{11}(x)}{\theta_{11}(t_2 x)\theta_{11}(x/t_1)} = \frac{1}{\theta_{11}(t_1 t_2)}, \tag{3.40}$$

(b) *and*

$$\frac{1}{\theta_{11}(t_1)\theta_{11}(t_2)} \oint_{\mathscr{C}_\varepsilon(q/t_2)} \frac{\mathrm{d}x}{2\pi i x} \frac{\theta_{11}(t_2 x/t_1)\theta_{11}(x)}{\theta_{11}(t_2 x)\theta_{11}(x/t_1)} = \frac{1}{\theta_{11}(t_1 t_2)}. \tag{3.41}$$

Proof. The proof of (b) is trivial once we prove (a). Note first that the two functions in (3.40) are multivalued. Since we can extract on both sides a term $\sqrt{t_1 t_2}$ we may think of (3.40) as an equality of single-valued functions.

$$\begin{aligned}
&\frac{1}{\theta_{11}(t_1)\theta_{11}(t_2)} \oint_{\mathscr{C}_3} \frac{\mathrm{d}x}{2\pi i x} \frac{\theta_{11}(t_2 x/t_1)\theta_{11}(x)}{\theta_{11}(t_2 x)\theta_{11}(x/t_1)} \\
&\quad = \frac{1}{\theta_{11}(t_1)\theta_{11}(t_2)} \\
&\quad\quad \times \oint_{\mathscr{C}_3} \frac{\mathrm{d}x}{2\pi i x(1-\frac{qt_1}{x})} \frac{\theta_{11}(t_2 x/t_1)\theta_{11}(x)(q)_\infty^2}{\theta_{11}(t_2 x)\prod_{i\geqslant 1}(1-\frac{q^{i+1}t_1}{x})(1-\frac{q^i x}{t_1})((x/t_1)^{1/2}-(x/t_1)^{-1/2})} \\
&\quad = \frac{\theta_{11}(qt_2)\theta_{11}(qt_1)(q)_\infty^2}{\theta_{11}(t_1)\theta_{11}(t_2)\theta_{11}(qt_1t_2)\prod_{i\geqslant 1}(1-q^{i+1})(1-q^i)(q^{1/2}-q^{-1/2})} \\
&\quad = \frac{\theta_{11}(t_2)\theta_{11}(t_1)(q)_\infty^2}{\theta_{11}(t_1)\theta_{11}(t_2)\theta_{11}(t_1t_2)(q)_\infty^2} = \frac{1}{\theta_{11}(t_1t_2)}. \qquad \square
\end{aligned} \tag{3.42}$$

Lemma 3.2 and (3.39) imply that the q-difference equation for $G(t_1,t_2)$ depends on the one-point function $G(t_1t_2)$, i.e.

$$G(qt_1,t_2) = -q^{1/2}t_1t_2(G(t_1,t_2)-G(t_1t_2)).$$

In the next section we will prove the same formula by using purely algebraic methods.

3.6. *q-difference equations for Bloch–Okounkov n-point functions*

In this part we analyze the difference equations for $G(t_1,\ldots,t_n)$ (at least when $n=2$) which gives us elliptic transformation formulas for the multi valued function $G(t_1,\ldots,t_n)$, i.e. transformation formulas for $G(qt_1,\ldots,t_n)$. If we put

$$H(t_1,\ldots,t_n,x_1,\ldots,x_n) = \frac{\prod_{i<j}\theta_{11}(t_jx_j/t_ix_i)\theta_{11}(x_j/x_i)}{\prod_{i<j}\theta_{11}(t_jx_j/x_i)\theta_{11}(x_j/t_ix_i)\prod_{i=1}^n\theta_{11}(t_i)},$$

then we have

$$H(qt_1,\dots,t_n,x_1,\dots,x_n) = -q^{1/2}t_1t_2\dots t_nH(t_1,\dots,t_n,x_1,\dots,x_n). \tag{3.43}$$

where we used the formula

$$\theta_{11}(qt) = -q^{-1/2}t^{-1}\theta_{11}(t).$$

We are aiming for a similar formula as in the case of $G(t_1,\dots,t_n)$. For simplicity let us consider first the 1 and 2-point functions.

We start with the 1-point function

$$G(t_1) = \eta(q)\,\mathrm{tr}|_{\mathscr{F}_0}X(\psi,t_1x_1)X(\psi^*,x_1)q^{\bar{L}(0)}. \tag{3.44}$$

Then we have

Proposition 3.3.

$$G(qt_1) + q^{1/2}t_1G(t_1) = \delta_{1/2}(qt_1). \tag{3.45}$$

Proof. First note that

$$[X(\psi,t_1x_1),X(\psi^*,x_1)] = \delta_{1/2}(t_1).$$

Also, we introduce a "control" variable y a $\mathbb{C}[y,y^{-1}]$-valued operator $y^{H(0)}$, where $H(0)$ is the charge operator introduced earlier.

$$\begin{aligned}
&\eta(q)\,\mathrm{tr}|_{\mathscr{F}}X(\psi,t_1x_1)X(\psi^*,x_1)y^{H(0)}q^{\bar{L}(0)}\\
&\quad= \eta(q)\,\mathrm{tr}|_{\mathscr{F}}\delta_{1/2}(t_1)y^{H(0)}q^{\bar{L}(0)} - \eta(q)\,\mathrm{tr}|_{\mathscr{F}}X(\psi^*,x_1)X(\psi,t_1x_1)y^{H(0)}q^{\bar{L}(0)}\\
&\quad= \eta(q)\,\mathrm{tr}_{\mathscr{F}}\delta_{1/2}(t_1)q^{\bar{L}(0)} - \mathrm{tr}|_{\mathscr{F}}\eta(q)X(\psi^*,x_1)y^{H(0)-1}q^{\bar{L}(0)}X\left(\psi,\frac{t_1x_1}{q}\right)\\
&\quad= \eta(q)\,\mathrm{tr}_{\mathscr{F}}\delta_{1/2}(t_1)q^{\bar{L}(0)} - \eta(q)\,\mathrm{tr}_{\mathscr{F}}X\left(\psi,\frac{x_1t_1}{q}\right)X(\psi^*,x_1)y^{H(0)-1}q^{\bar{L}(0)}.
\end{aligned} \tag{3.46}$$

Now if we extract the coefficient of y^0 in (3.46) we obtain

$$G(t_1) = \delta_{1/2}(t_1) - q^{-1/2}t_1G(t_1/q).$$

After multiplying with $q^{D_{t_1}}$ we obtain the desired formula. □

Note that (3.45) is a formal analogue of the difference equation for one-point functions obtained in [3]. By using the simple identity $(1-t)\delta_{1/2}(t) = 0$. we obtain

$$(1-qt_1)(G(qt_1) + q^{1/2}t_1G(t_1)) = 0.$$

Note that in the equation above we cannot remove the term $(1-qt_1)$ because this is a formal expression. On the other hand this equation gives us meromorphic continuation on (a double cover of) $\mathbb{C}^{\times}$ (provided that $G(t_1)$ is meromorphic in Ω and such that $q^{\mathbb{Z}} \cdot \Omega = \mathbb{C}^{\times}$). Now we consider transition from the formal variables to the complex numbers. From (3.45) it follows

$$\begin{aligned} G(qt_1) + q^{1/2} t_1 G(t_1) &= (q^{D_{t_1}} + q^{1/2} t_1) G(t_1) \\ &= (1 + q^{D_{t_1}} q^{-1/2} t_1^{-1}) q^{1/2} t_1 G(t_1) \\ &= \delta_{1/2}(qt_1). \end{aligned} \tag{3.47}$$

Now we formally invert the operator $(1 + q^{D_{t_1}} q^{-1/2} t_1^{-1})$ by using a formal nexpansion

$$\frac{1}{1+A} = \sum_{n \geqslant 0} (-1)^n A^n.$$

Then we have

$$\begin{aligned} q^{1/2} t_1 G(t_1) &= \frac{1}{1 + q^{D_{t_1}} q^{-1/2} t_1^{-1}} \, \delta_{1/2}(qt_1) \\ &= \frac{1}{1 + q^{-3/2} t_1^{-1} q^{D_{t_1}}} \, \delta_{1/2}(qt_1). \end{aligned} \tag{3.48}$$

Therefore

$$\begin{aligned} G(t_1) &= q^{-1/2} t_1^{-1} \sum_{m \geqslant 0} (-1)^m (q^{-3/2} t_1^{-1} q^{D_{t_1}})^m \delta_{1/2}(qt_1) \\ &= q^{-1/2} t_1^{-1} \sum_{n \in \mathbb{Z}+1/2} \sum_{m \geqslant 0} (q^{-3/2} t_1^{-1} q^{D_{t_1}})^m q^n t_1^n \\ &= \sum_{k \in \mathbb{Z}} \sum_{m \geqslant 0} (-1)^m q^{m(m+1)/2} q^{-k(m+1)} t_1^{-k-1/2}, \end{aligned} \tag{3.49}$$

where $n - m = -k + 1/2$. Now, since

$$\frac{m(m+1)}{2} - k(m+1) = \frac{(m-k+1/2)^2}{2} - \frac{(k+1/2)^2}{2},$$

we obtain

$$G(t_1) = \sum_{k \in \mathbb{Z}} t_1^{-k-1/2} q^{-(k+1/2)^2/2} \sum_{m \geqslant 0} (-1)^m q^{(m-k+1/2)^2/2}. \tag{3.50}$$

By using the symmetry on the right-hand side of (3.50) and after summing the geometric series we obtain

$$G(t_1) = t_1^{-1/2} \sum_{m \geqslant 0} (-1)^m q^{m(m+1)/2} \left(\frac{q^m t_1^{-1}}{1 - t_1^{-1} q^m} + 1 + \frac{q^{m+1} t_1}{1 - t_1 q^{m+1}} \right). \tag{3.51}$$

If we switch to complex variables then the Note that (3.51) is a convergent series when $1 < |t| < 1/|q|$, and it can be analytically extended (by using (3.51)) to all values

$t \neq q^n$, $n \in \mathbb{Z}$. Thus

$$G(t_1) = t_1^{-1/2} \sum_{m \geqslant 0} (-1)^m q^{m(m+1)/2} \, \frac{1 - q^{2m+1}}{(1 - t_1^{-1} q^m)(1 - t_1 q^{m+1})}. \tag{3.52}$$

We claim that $G(t_1)$ in (3.52) is equal to (3.44) (remember we just find a solution of (3.3)). For this apply Lemma A.2 from Appendix A.

Now, if we combine formula (3.52) with Corollary (3.1) we see that the reciprocal theta function can be written as an infinite sum (with the same set of poles). The similar formula follows from the denominator formula for $N = 2$ superconformal algebra.

Now we discuss two point functions. Again the aim is to derive q-difference equation for the two point function by using a formal variable approach.

Proposition 3.4. *We have*

$$G(qt_1, t_2) + q^{1/2} t_1 t_2 G(t_1, t_2) = \delta_{1/2}(qt_1) G(t_2) + q^{1/2} t_1 t_2 G(t_1 t_2). \tag{3.53}$$

Moreover we have a locality equation:

$$(1 - qt_1t_2)(1 - qt_1)\{G(qt_1, t_2) + q^{1/2} t_1 t_2 G(t_1, t_2) - q^{1/2} t_1 t_2 G(t_1 t_2)\} = 0. \tag{3.54}$$

Proof. The proof is very similar as the one in Proposition (3.3). So we omit details. By using the Jacobi identity

$$\begin{aligned}
&\mathrm{tr}|_{\mathscr{F}} \eta(q) \mathrm{o}(X(\psi, t_1 x_1) X(\psi^*, x_1)) \mathrm{o}(X(\psi, t_2 x_2) X(\psi^*, x_2)) y^{H(0)} q^{\bar{L}(0)} \\
&\quad + \mathrm{tr}|_{\mathscr{F}} \eta(q) \mathrm{o}\left(X\left(\psi, \frac{t_1 x_1}{q}\right) X(\psi^*, x_1)\right) \mathrm{o}(X(\psi, t_2 x_2) X(\psi^*, x_2)) y^{H(0)-1} q^{\bar{L}(0)} \\
&\qquad = \mathrm{tr}|_{\mathscr{F}} \eta(q) \delta_{1/2}(t_1) X(\psi, t_2 x_2) X(\psi^*, x_2) y^{H(0)} q^{\bar{L}(0)} \\
&\qquad\quad - \mathrm{tr}|_{\mathscr{F}} X(\psi, t_1 t_2 x_1) X(\psi^*, x_1) y^{H(0)} q^{\bar{L}(0)} + \mathrm{tr}_{\mathscr{F}} \delta_{1/2}(t_1 t_2) y^{H(0)} q^{\bar{L}(0)}.
\end{aligned} \tag{3.55}$$

If we extract coefficient of $(x_1 x_2)^0$ in (3.55), by using Theorem 3.1, we obtain

$$\begin{aligned}
&G(t_1, t_2) + q^{-1/2} t_1 t_2 G\left(\frac{t_1}{q}, t_2\right) \\
&\quad = \delta_{1/2}(t_1) G(t_2) - G(t_1 t_2) + \delta_{1/2}(t_1 t_2).
\end{aligned} \tag{3.56}$$

If we act with q^{D_t} we obtain

$$\begin{aligned}
&G(qt_1, t_2) + q^{1/2} t_1 t_2 G(t_1, t_2) \\
&\quad = q^{1/2} t_1 t_2 G(t_1 t_2) - \delta_{1/2}(qt_1 t_2) + \delta_{1/2}(qt_1) G(t_2) + \delta_{1/2}(qt_1 t_2) \\
&\quad = q^{1/2} t_1 t_2 G(t_1 t_2) + \delta_{1/2}(qt_1) G(t_2). \qquad \square
\end{aligned} \tag{3.57}$$

By using this method we can derive similar formulas for an arbitrary n-point function. A similar theorem has been proven in [3] by using different methods.

Theorem 3.2. *Formally*
(a)

$$G(t_1,\ldots,t_n)+q^{-1/2}\left(\prod_{i=1}^{n} t_i\right)G\left(\frac{t_1}{q},t_2,\ldots,t_n\right)$$

$$=-\sum_{s=1}^{n-1}\sum_{1<t_{i_1}<\cdots<t_{i_s}\leqslant n} G(t_1t_{i_1}\cdots t_{i_s},\ldots,\hat{t}_{i_1},\ldots,\hat{t}_{i_k},\ldots)$$

$$+\delta_{1/2}(t_{i_1}\cdots t_{i_s})G(\cdots\hat{t}_{i_1}\cdots\hat{t}_{i_s}\cdots).+\delta_{1/2}(t_1)G(t_2,\ldots,t_n). \tag{3.58}$$

(b)

$$\prod_{0\leqslant s\leqslant n-1, 1<i_1<\cdots i_s\leqslant n}(1-qt_1t_{i_1}\ldots t_{i_s})\left(G(qt_1,\ldots,t_n)+q^{1/2}\prod_{i=1}^{n}t_i\sum_{s=0}^{n-1}\right.$$

$$\left.\sum_{1<i_1<\cdots<i_s\leqslant n}(-1)^sG(t_1t_{i_1}\cdots t_{i_s},\ldots\hat{t}_{i_1},\ldots,\hat{t}_{i_s},\ldots)\right)=0. \tag{3.59}$$

(c) *Let* $|q|<1$. *Then* $G(t_1,\ldots,t_n)$ *converges inside an open subset*

$$\Omega_{n,0}\cap\left\{(t_1,\ldots,t_n)\colon \left|\prod_{i=1}^{n}t_i\right|<\frac{1}{q}\right\},$$

and it has a meromorphic continuation on (*a double cover of*) $(\mathbb{C}^{\times})^n$ *with the set of poles contained in* $\{(t_1,\ldots,t_n)\colon q^mt_{i_1}\ldots t_{i_n}=1,\ 1\leqslant i_1<\cdots<i_s\leqslant n, m\in\mathbb{Z}\}$.

Proof. The idea is essentially the same as in (3.55). We prove (a) directly and use the induction for (b). Actually (b) follows if

$$G(qt_1,\ldots,t_n)+q^{1/2}\prod_{i=1}^{n}t_iG(t_1,\ldots,t_n)$$

$$=q^{1/2}\prod_{i=1}^{n}t_i\sum_{s=1}^{n-1}\sum_{1<i_1<\cdots<i_s\leqslant n}(-1)^{s-1}G(t_1t_{i_1}\cdots t_{i_s},\ldots,\hat{t}_{i_1},\ldots,\hat{t}_{i_s}\ldots)$$

$$+\sum_k\sum_{j_1,\ldots,j_k}\alpha_{j_1,\ldots,j_k}\delta(qt_{j_1}\cdots t_{j_k})G(\ldots,\hat{t}_{j_1},\ldots\hat{t}_{j_k},\ldots), \tag{3.60}$$

where $\alpha_{j_1,\ldots,j_k}$ are some constant. By multiplying (3.60) by

$$\prod_{\substack{0\leqslant s\leqslant n-1\\ 1<i_1<\cdots<i_s\leqslant n}}(1-qt_1t_{i_1}\cdots t_{i_s})$$

we get (3.60).

For $n=1$ and $n=2$ (3.60) holds. Suppose that it holds for every $k \leqslant n-1$. Then

$$\sum_{s=1}^{k} \sum_{1<i_1<\cdots<i_s\leqslant k} \alpha_s G(qt_1t_{i_1}\cdots t_{i_s},\ldots,\hat{t}_{i_1},\ldots,\hat{t}_{i_s},\ldots)$$
$$=\sum_{r=1}^{k}(-1)^{r-1}\sum_{1<j_1<\cdots<j_r\leqslant k}\sum_{m=1}^{r}\binom{r}{m}(-1)^m\alpha_m G(t_1t_{i_1}\cdots t_{i_r},\ldots,\hat{t}_{i_1},\ldots\hat{t}_{i_r},\ldots)$$
$$+\cdots\ . \tag{3.61}$$

Where dots stand for terms that involve $\delta(qt_{i_1}\cdots t_{i_k})$. In particular if for every i, $\alpha_i=1$ then

$$\sum_{m=1}^{r}\binom{r}{m}(-1)^m\alpha_m=-1,$$

and

$$\sum_{s=1}^{k}\sum_{i_1<\cdots<i_s\leqslant k} G(qt_1t_{i_1}\cdots t_{i_s},\ldots,\hat{t}_{i_1},\ldots,\hat{t}_{i_s},\ldots)$$
$$=-\sum_{r=1}^{k}(-1)^{r-1}\sum_{1<j_1<\cdots<j_r\leqslant k} G(t_1t_{i_1}\cdots t_{i_r},\ldots,\hat{t}_{i_1},\ldots,\hat{t}_{i_r}\ldots)+\cdots \tag{3.62}$$

Let o stands for the operator $\mathrm{Coff}_{x_1^0\cdots x_n^0}$.

$$\mathrm{tr}|_{\mathscr{F}}\mathrm{o}(X(\psi,x_1t_1)X(\psi^*,t_1)\cdots X(\psi,x_nt_n)X(\psi^*,t_n))q^{\bar{L}(0)}y^{H(0)}$$
$$=\mathrm{tr}|_{\mathscr{F}}\mathrm{o}([X(\psi,x_1t_1),X(\psi^*,x_1)\cdots X(\psi,x_nt_n)X(\psi^*,x_n)])q^{\bar{L}(0)}y^{H(0)}$$
$$-\mathrm{tr}|_{\mathscr{F}}\mathrm{o}(X(\psi^*,x_1)\cdots X(\psi,x_nt_n)X(\psi^*,x_n)X(\psi,x_1t_1))q^{\bar{L}(0)}y^{H(0)}. \tag{3.63}$$

Therefore

$$\mathrm{tr}|_{\mathscr{F}}\mathrm{o}(X(\psi,x_1t_1)X(\psi^*,x_1)\cdots X(\psi,t_nx_n)X(\psi^*,x_n))q^{\bar{L}(0)}y^{H(0)}$$
$$+\mathrm{tr}|_{\mathscr{F}}\mathrm{o}\left(X\left(\psi,\frac{x_1t_1}{q}\right)X(\psi^*,x_1)\cdots X(\psi,x_nt_n)X(\psi^*,x_n)\right)q^{\bar{L}(0)}y^{H(0)-1}$$
$$=\mathrm{tr}|_{\mathscr{F}}\mathrm{o}(\delta_{1/2}(t_1)X(\psi,t_2x_2)X(\psi^*,x_2)\ldots X(\psi,t_nx_n)X(\psi^*,x_n))q^{\bar{L}(0)}y^{H(0)}$$
$$+\mathrm{tr}|_{\mathscr{F}}\mathrm{o}(X(\psi^*,x_1)X(\psi,t_2x_2)\delta_{1/2}\left(\frac{t_1x_1}{x_2}\right)$$

$$\cdots X(\psi, t_n x_n) X(\psi^*, x_n)) q^{\bar{L}(0)} y^{H(0)} + \cdots$$
$$+ \mathrm{tr}\, \mathrm{o}\left(X(\psi^*, t_1) \cdots X(\psi, t_n x_n) \delta_{1/2}\left(\frac{t_1 x_1}{x_n}\right)\right) q^{\bar{L}(0)} y^{H(0)}. \tag{3.64}$$

Now in (3.64) we use a formal delta function substitution property (cf. [10])

$$X(u, t_2 x_2) \delta_{1/2}\left(\frac{t_1 x_1}{x_2}\right) = X(u, t_1 t_2 x_1) \delta\left(\frac{t_1 x_1}{x_2}\right). \tag{3.65}$$

This fact was used in Part I [24] as well. Then

$$\mathrm{tr}|_{\mathscr{F}} \mathrm{o}(X(\psi, t_1 x_1) X(\psi^*, x_1) \cdots X(\psi, t_n x_n) X(\psi^*, x_n)) q^{\bar{L}(0)} y^{H(0)}$$
$$+ \mathrm{tr}|_{\mathscr{F}} \mathrm{o}\left(X\left(\psi, \frac{t_1 x_1}{q}\right) X(\psi^*, x_1) \cdots X(\psi, t_n x_n) X(\psi^*, x_n)\right) q^{\bar{L}(0)} y^{H(0)-1}$$
$$= \mathrm{tr}|_{\mathscr{F}} \mathrm{o}(\delta_{1/2}(t_1) X(\psi, t_2 x_2) X(\psi^*, x_2) \cdots X(\psi, t_n x_n) X(\psi^*, x_n)) q^{\bar{L}(0)} y^{H(0)}$$
$$+ \mathrm{tr}|_{\mathscr{F}} \mathrm{o}(X(\psi^*, x_1) X(\psi, t_1 t_2 x_1) \cdots X(\psi, t_n x_n) X(\psi^*, x_n)) q^{\bar{L}(0)} y^{H(0)} + \cdots$$
$$+ \mathrm{tr}|_{\mathscr{F}} \mathrm{o}(X(\psi^*, x_1) X(\psi, t_2 x_2) \cdots X(\psi, t_1 t_n x_1)) q^{\bar{L}(0)} y^{H(0)}. \tag{3.66}$$

We move the term $X(\psi, t_1 t_i x_1)$, $i = 2, \ldots, n$ to the right-hand side of (3.66) to the left by anticommuting them with operators $X(\psi^*, x_j)$, $j \leqslant i$. This anticommuting produces more terms, etc. If we repeat procedure of anticommuting and delta function substitution for each term (except the first term) on the right-hand side of (3.66) we obtain

$$\mathrm{tr}|_{\mathscr{F}} \mathrm{o}(X(\psi^*, x_1) X(\psi, t_1 t_2 x_1) \cdots X(\psi, t_n x_n) X(\psi^*, x_n)) q^{\bar{L}(0)} y^{H(0)}$$
$$= \mathrm{tr}|_{\mathscr{F}} \delta_{1/2}(t_1 t_2) \mathrm{o}(\cdots X(\psi, t_n x_n) X(\psi^*, x_n)) q^{\bar{L}(0)} y^{H(0)}$$
$$- \mathrm{tr}|_{\mathscr{F}} \mathrm{o}(X(\psi, t_1 t_2 x_1) X(\psi^*, x_1) \cdots X(\psi, t_n x_n) X(\psi^*, x_n)) q^{\bar{L}(0)} y^{H(0)}, \tag{3.67}$$

for the second term,

$$\mathrm{tr}|_{\mathscr{F}} \mathrm{o}(X(\psi^*, x_1) X(\psi, t_2 x_2) \cdots X(\psi, t_1 t_i x_1) \cdots X(\psi, t_n x_n) X(\psi^*, x_n)) q^{\bar{L}(0)} y^{H(0)}$$
$$= -\sum_{r=1}^{i-1} \sum_{1 < t_{s_1} < \cdots < t_{s_r} \leqslant i} \mathrm{tr}|_{\mathscr{F}} \mathrm{o}(X(\psi, t_{s_1} \cdots t_{s_r} t_i x_1) X(\psi^*, x_1) \cdots$$
$$\cdots X(\psi, t_n x_n) X(\psi^*, x_n)) q^{\bar{L}(0)} y^{H(0)} + \mathrm{tr}|_{\mathscr{F}} \delta_{1/2}(t_1 \cdots \hat{t}_{s_1} \cdots \hat{t}_{s_r} \cdots t_i)$$
$$\times \mathrm{o}(X(\psi, t_{s_1} x_{s_1}) X(\psi^*, x_{s_1}) \cdots X(\psi, t_{s_r} x_{s_r}) \cdots X(\psi^*, x_{s_r})) q^{\bar{L}(0)} y^{H(0)} \tag{3.68}$$

for the $i+1$th term, etc. Combining (3.66)–(3.68) we obtain

$$G(t_1, \ldots, t_n) + q^{-1/2} \prod_{i \geqslant 1}^{n} t_i G\left(\frac{t_1}{q}, t_2, \ldots, t_n\right)$$

$$= -\sum_{s=1}^{n-1} \sum_{1<t_{i_1}<\cdots<t_{i_s}} G(t_1 t_{i_1} \cdots t_{i_s}, \ldots, \hat{t}_{i_1}, \ldots, \hat{t}_{i_k}, \ldots)$$

$$+ \delta_{1/2}(t_{i_1} \cdots t_{i_s}) G(\ldots, \hat{t}_{i_1}, \ldots \hat{t}_{i_s} \ldots). \tag{3.69}$$

This is exactly (3.58). Now if we act by $q^{D_{t_1}}$ on (3.69) we obtain

$$G(qt_1, \ldots, t_n) + q^{1/2} t_1 \ldots t_n G(t_1, t_2, \ldots, t_n)$$

$$= -\sum_{s \geqslant 1} \sum_{1<t_{i_1}<\cdots<t_{i_s}} G(qt_1 t_{i_1} \ldots t_{i_s}, \cdots, \hat{t}_{i_1}, \cdots, \hat{t}_{i_k}, \cdots)$$

$$+ \delta(qt_{i_1} \ldots t_{i_s}) G(\ldots \hat{t}_{i_1} \ldots \hat{t}_{i_s} \ldots). \tag{3.70}$$

If we apply now (3.62) and the inductive assumption we obtain

$$G(qt_1, \ldots, t_n) + q^{1/2} t_1 \ldots t_n G(t_1, t_2, \ldots, t_n)$$

$$= q^{1/2} \prod_{i=1}^{n} t_i \sum_{s \geqslant 1} \sum_{1<i_1<\cdots<i_s \leqslant n} (-1)^{s-1} G(t_1 t_{i_1} \ldots t_{i_s}, \ldots \hat{t}_{i_1} \ldots \hat{t}_{i_s} \ldots)$$

$$+ \delta(qt_{i_1} \ldots t_{i_s}) G(\ldots \hat{t}_{i_1} \ldots \hat{t}_{i_s} \ldots) + \cdots \tag{3.71}$$

which is (3.60). There are essentially two ways of proving (c). First which uses an explicit q-expansion for $G(t_1, \ldots, t_n)$ and the second that uses Theorem 3.1. The first method is much harder though. Let us explain the second proof. The crucial observation is that

$$\Upsilon_n \cap \Omega_{n,0} \subset \Lambda_n(x) \tag{3.72}$$

holds for every x where

$$\Upsilon_n = \left\{ (t_1, \ldots, t_n)\colon \left| \prod_{i=1}^{n} t_i \right| < \frac{1}{q} \right\}.$$

This is easy to prove by induction. Since the $2n$-point function converges absolutely inside $\Omega_{n,n}$ and it has an expansion in terms of x_i's, all terms converge absolutely, hence the zeroth term $G(t_1, \ldots, t_n)$ converges absolutely inside $\Upsilon_n \cap \Omega_{n,0}$. Analytic extension and information about poles follows from the part (b) by using the induction. □

Now let us try to solve (3.56) for $n = 2$. For $n \geqslant 3$ it is possible to obtain similar explicit formula. Because, the expansion in t_i's converges in a tiny domain, i.e., $\Upsilon_n \cap \Omega_{n,0}$, these formulas are hard to express in a nice way (meaning, that the poles and domain of convergence are visible). On the contrary, product formulas have very nice form (cf. [3]).

Proposition 3.5. (a) (3.56) *has the unique solution in* $\mathbb{C}[[q, t_1^{\pm 1}, t_2^{\pm 1}]]$ *that satisfy* $G(t_1, t_2) = G(t_2, t_1)$ *and it is given by*

$$\begin{aligned}
&G(t_1,t_2)\\
&\quad = (t_1t_2)^{-1/2}\sum_{m\geqslant 0}(-1)^m q^{\frac{m(m+1)}{2}}\frac{1-q^{2m+1}}{(1-(t_1t_2)^{-1}q^m)(1-(t_1t_2)q^{m+1})}\\
&\qquad + (t_1t_2)^{-1/2}\sum_{(k,l)\in\mathbb{Z}^2, k\neq l}\frac{t_1^{-l}t_2^{-k}}{1-q^{|l-k|}}\left(\sum_{m=0}^{\infty}(-1)^{m+1}q^{\frac{m(m+1+|\min(k,l)|)}{2}}\right.\\
&\qquad \left.-q^{|l-k|}\sum_{m=0}^{\infty}(-1)^{m+1}q^{\frac{m(m+1+|\max(k,l)|)}{2}}\right).
\end{aligned}$$

(b) *Let* $|q| < 1$. *Then* $G(t_1, t_2)$ *absolutely converges inside*

$$\Omega_{2,0} \cap \left\{(t_1,t_2)\colon\ |t_1t_2| < \frac{1}{|q|}\right\}.$$

Moreover, it has a meromorphic extension to a double cover of $(\mathbb{C}^\times)^2$ *with the set of poles contained in*

$$(\mathbb{C}^\times)^2 \setminus \{(t_1,t_2)\colon\ t_1t_2q^m = 1,\ t_1q^m = 1,\ t_2q^m = 1,\ m\in\mathbb{Z}\}.$$

Proof. The fact about the uniqueness follows from Lemma A.4 from the Appendix A. Proof of part (a) is quite lengthy, but straightforward and essentially uses the same procedure as in the case of 1-point function. Here is the main idea. Formula (3.56) can be written as

$$q^{1/2}t_1t_2G(t_1,t_2) = \frac{1}{1+q^{-3/2}t_1^{-1}t_2^{-1}q^{D_{t_1}}}(\delta_{1/2}(qt_1)G(t_2) + q^{1/2}t_1t_2G(t_1t_2)). \tag{3.73}$$

The fraction in the above formula can be expanded as in the $n = 1$ case. The rest is manipulations with q-series.

We can prove part (b) in two different ways. The fact about the convergence of $G(t_1, t_2)$ follows either from the previous theorem (which holds for every n) or by using explicit expression (3.73). Hence by using this fact we can meromorphically extend (3.56) to remaining values of t by using q-difference for $G(qt_1, t_2)$ and $G(q^{-1}t_1, t_2)$ and then by using $G(t_1, qt_2)$ and $G(t_1, q^{-1}t_2)$ (which is obtained from (3.56) because $G(t_1, t_2) = G(t_2, t_1)$). From the q-difference equation it follows that we cannot extend the function to those values for which either $G(t_1t_2)$ has poles or $G(t_1)$ and $G(t_2)$ hence the statement follows. □

It is worth mentioning that (as in the $n = 1$ case) we encounter certain q-series that have singular parts consisting of single terms in the q-expansion. Once we combined several of these series all singular terms cancel. This should be compared with some results from Part I [24]. Let us illustrate this using a particular example. An *incomplete* θ-function is an expression of the form

$$A_k(q) = \sum_{m=0}^{\infty} (-1)^m q^{(m+1-2k)(m+2)/2},$$

for some $k \in \mathbb{Z}$. It is clear that $A_k(q) \in \mathbb{Z}[[q]]$ for $k \leqslant 0$. Also it is not hard to show that

$$A_k(q) = \frac{1}{q^k} + \sum_{m=0}^{\infty} (-1)^{m+1} q^{m(m+2k+1)/2} \in \frac{1}{q^k} + \mathbb{Z}[[q]],$$

for $k > 0$. Then

$$A_{\min(k,l)}(q) - q^{|l-k|} A_{\max(k,l)}(q) \in \mathbb{Z}[[q]],$$

for every choice of constants $l, k \in \mathbb{Z}$, $k \neq l$. This fact is used in proving (3.73). Also notice that coefficients of $t_1^{-k-1/2} t_2^{-l-1/2}$ in (3.73) are given by

$$A_{k,l} = \sum_{m,n \geqslant 0} (-1)^{m+n} q^{(n+m+1)(n+m+2)/2 - k(m+1) - l(n+1)} \in \mathbb{Z}[[q]],$$

for every $k, l \in \mathbb{Z}$, $k \neq l$. Some related q-series were studied in [15].

3.7. General case

So far we only discussed charge 0 subspace, i.e. $\mathscr{F}_0 \subset \mathscr{F}$. However $\mathscr{F}_m$ is also a $\hat{\mathscr{D}}$-module. In vertex operator algebra language, $\mathscr{F}_m$ is an irreducible $\mathscr{F}_0$-module ($M(1) \cong \mathscr{F}_0 \cong L_1$ cf. [1]). Let us denote by $H_m(t_1, \ldots, t_n, x_1, \ldots, x_n)$ the restriction of (3.23) to the space $\mathscr{F}_m$ and with $G_m(t_1, \ldots, t_n)$ the corresponding Bloch–Okounkov n-point function. For every $m \in \mathbb{Z}$ we have

$$H_m(qt_1, \ldots, t_n, x_1, \ldots, x_n) = -q^{n+1/2} t_1 t_2 \cdots t_n H_m(t_1, \ldots, t_n, x_1, \ldots, x_n).$$

Also,

$$G_m(t_1, \ldots, t_n) = q^{n^2/2} t_1^n \cdots t_2^n G(t_1, \ldots, t_n).$$

Hence the "total" Bloch–Okounkov n-point function obtained by taking the trace over $\mathscr{F}$, and it is given by

$$\sum_{m \in \mathbb{Z}} q^{m^2/2} \left(\prod_{i=1}^{n} t_i \right)^m G(t_1, \ldots, t_n) = \theta \left(\prod_{i=1}^{m} t_i \right) G(t_1, \ldots, t_n).$$

4. q-traces for $\mathscr{D}^-$; Majorana fermion case

4.1. Iterated $2n$-point functions

Let us recall some notation from first part [24]. Denote by F the Fock fermionic space associated to a free (or Majorana) fermion. Let $\varphi = \varphi(-1/2)\mathbf{1} \in F$, such that

$$X(\varphi,x)=\sum_{n\in\mathbb{Z}}\varphi_n x^{-n-1/2}$$

and

$$[\varphi_m,\varphi_n]=\delta_{m+n,0}.$$

The aim is to study q-graded traces of the form:

$$\mathrm{tr}|_{\mathscr{F}}X(Y[\varphi,y_1]\varphi,x_1)\cdots X(Y[\varphi,y_n]\varphi,x_n)q^{\bar{L}(0)}. \tag{4.1}$$

If we modify Corollary 2.2 and Remark 2.2, such that it applies in the vertex operator superalgebra case it follows that all the information about (4.1) can be obtained by studying the following $2n$-point function:

$$\mathrm{tr}|_F X(\varphi,t_1x_1)X(\varphi,x_1)\cdots X(\varphi,t_nx_n)X(\varphi,x_n)q^{\bar{L}(0)}. \tag{4.2}$$

Again, once we switch to the complex variables (4.2) is multi-valued with respect to t_i variables.

It is well-known that one can obtain a free fermion from a pair of charged fermions. Let $\tilde{\varphi}(n)=\psi(n)+\psi^*(n)/\sqrt{2}$, for $n\in\mathbb{Z}+\frac{1}{2}$. Then

$$[\tilde{\varphi}(m),\tilde{\varphi}(n)]=\delta_{m+n,0},$$

hence computing (4.1), up to the character, reduces to the case of correlation functions with charged fermions. Hence, it is not hard to see that the correlation function (4.2) converges inside

$$|t_1x_1|>|x_1|>\cdots>|x_n|>|qt_1x_1|>0,$$

to a (multivalued) analytic function. Moreover, (4.2) has a meromorphic continuation to a double cover of $(\mathbb{C}^\times)^{2n}$. Another way of proving convergence, is by explicit calculations.

We present the calculations for $n=2$.

$$\begin{aligned}
&\mathrm{tr}|_F X(\varphi,t_1x_1)X(\varphi,x_1)X(\varphi,t_2x_2)X(\varphi,x_2)q^{\bar{L}(0)}\\
&\quad+\mathrm{tr}|_F X(\varphi,t_1x_1)X(\varphi,x_1)\cdots X(\varphi,t_nx_n)X(\varphi,x_n)q^{\bar{L}(0)}\\
&\qquad=\mathrm{tr}|_F[X(\varphi,t_1x_1),X(\varphi,x_1)\cdots X(\varphi,t_nx_n)X(\varphi,x_n)]q^{\bar{L}(0)}\\
&\qquad=\mathrm{tr}|_F\delta_{1/2}\left(\frac{1}{t_1}\right)X(\varphi,t_2x_2)X(\varphi,x_2)q^{\bar{L}(0)}\\
&\qquad\quad-\mathrm{tr}|_F\delta_{1/2}\left(\frac{t_2x_2}{t_1x_1}\right)X(\varphi,x_1)X(\varphi,x_2)q^{\bar{L}(0)}\\
&\qquad\quad+\mathrm{tr}|_F\delta_{1/2}\left(\frac{x_2}{t_1x_1}\right)X(\varphi,t_1)X(\varphi,t_2x_2)q^{\bar{L}(0)}.
\end{aligned} \tag{4.3}$$

Hence we get

$$\begin{aligned}&\mathrm{tr}|_F X(\varphi,t_1x_1)X(\varphi,x_1)\cdots X(\varphi,t_nx_n)X(\varphi,x_n)q^{\bar{L}(0)}\\&\quad= G(t_1)G(t_2)\,\mathrm{tr}|_F q^{\bar{L}(0)} - G(t_1x_1/t_2x_2)G(x_1/x_2)\,\mathrm{tr}|_F q^{\bar{L}(0)}\\&\qquad+ G(t_1x_1/x_2)G(t_1/t_2x_2)\,\mathrm{tr}_F q^{\bar{L}(0)},\end{aligned} \tag{4.4}$$

where $G(t)$ is defined as in (3.50) and

$$\mathrm{tr}|_F q^{\bar{L}(0)} = \frac{\eta(q)^2}{\eta(q^2)\eta(q^{1/2})}.$$

As before we consider and the corresponding Bloch–Okounkov n-point function

$$\begin{aligned}&D(t_1,\dots,t_n)\\&\quad:= \frac{\eta(q^2)\eta(q^{1/2})}{\eta(q)^2}\,\mathrm{tr}|_F \mathrm{o}(X(\varphi,t_1x_1)X(\varphi,x_1))\cdots \mathrm{o}(X(\varphi,t_nx_n)X(\varphi,x_n))q^{\bar{L}(0)}.\end{aligned} \tag{4.5}$$

4.2. *q-difference equations for n-point functions*

Here we obtain q-difference equations for (4.5).

Theorem 4.1.

(a)

$$\begin{aligned}D(t_1,\dots,t_n) &= -D\left(\frac{t_1}{q},t_2,\dots,t_n\right) + \delta_{1/2}(t_1)D(t_2,\dots,t_n)\\&\times \sum_{s=1}^{n-1}\sum_{1<i_1<\cdots<i_s\leqslant n}\sum_{\varepsilon_{i_1},\dots,\varepsilon_{i_s}\in\{-1,1\}} D(t_1t_{i_1}^{\varepsilon_{i_1}}\cdots t_{i_s}^{\varepsilon_{i_s}},\dots,\hat{t}_{i_1},\dots\hat{t}_{i_s},\dots)\\&+\delta_{1/2}(t_{i_1}\cdots t_{i_s})D(\dots,\hat{t}_{i_1},\dots,\hat{t}_{i_s},\dots).\end{aligned} \tag{4.6}$$

(b)

$$\begin{aligned}&\prod_{s,i_1,\dots,i_s;\varepsilon_{i_1},\dots,\varepsilon_{i_s}\in\{-1,1\}}(1-qt_{i_1}^{\varepsilon_{i_1}}\dots t_{i_s}^{\varepsilon_{i_1}})\Bigg(D(qt_1,\dots,t_n)\\&\quad+\sum_{s=0}^{n-1}\sum_{1<i_1,\dots,i_s\leqslant n}\sum_{\varepsilon_{i_1},\dots,\varepsilon_{i_s}\in\{-1,1\}} D(t_1t_{i_1}^{\varepsilon_{i_1}}\cdots t_{i_s}^{\varepsilon_{i_s}},\dots,\hat{t}_{i_1},\dots,\hat{t}_{i_s},\dots)\Bigg)=0.\end{aligned} \tag{4.7}$$

(c) *$D(t_1,\dots,t_n)$ converges inside an open subset of $\Omega_{n,0}$, and it has a meromorphic continuation on a double cover of $(\mathbb{C}^\times)^n$ with the set of poles*

$$\{t_{i_1}^{\varepsilon_1}\dots t_{i_s}^{\varepsilon_s}q^m=1, 1\leqslant i_1<\cdots<i_n\leqslant n\in\mathbb{Z}_{>0}, m\in\mathbb{Z}, \varepsilon_j\in\{-1,1\}\}.$$

Proof. Part (a) is essentially the same as the proof of Theorem 3.2. The only difference are factors $\varepsilon_i \in \{1,-1\}$. Part (b) follows directly from (a) by acting with $q^{D_{t_1}}$. Part (c) follows from (a) and a similar argument as in Theorem 3.2. $\square$

5. q-traces for $\hat{\mathscr{D}}^+$; bosonic case

5.1. 1- and 2-point functions

For $k \geqslant 2$, let us define a normalized kth Eisenstein series

$$\tilde{G}_{2k}(q) = \frac{-B_{2k}}{(2k)!} + \frac{2}{(2k-1)!}\sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n \in \mathbb{C}[[q]],$$

where σ_k is the sum of the k-powers of the divisors of n and B_{2k} a Bernoulli number. If $q = e^{2\pi i \tau}$ we will write $\tilde{G}_{2k}(\tau)$ instead of $\tilde{G}_{2k}(q)$. Also we define the meromorphic functions

$$\zeta(z,\tau) = \frac{1}{z} + \sum_{\omega \in \mathbb{Z}+\mathbb{Z}\tau} \frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2},$$

$$\wp_2(z,\tau) = \frac{1}{z^2} + \sum_{\omega \in \mathbb{Z}+\mathbb{Z}\tau} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

and

$$\wp_{k+1}(z,\tau) = -\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}z}\wp_k(z,\tau),$$

for $k \geqslant 2$. Hence

$$(-1)^k (k-1)!\,\wp_k(z,\tau) = \left(\frac{\partial}{\partial z}\right)^{k-2} \wp_2(z,\tau).$$

$\zeta(z,\tau)$ is the Weierstrass zeta-function, with the property:

$$\zeta(z+\tau,\tau) = \zeta(z,\tau) + 2\zeta\left(\frac{\tau}{2},\tau\right).$$

For $k \geqslant 2$, $\wp_k$ is elliptic functions; $\wp_2(z,\tau)$ is usually called the Weierstrass $\wp$-function. In the annulus

$$0 < |z| < \min_{m,n} |m + n\tau|,$$

$\wp_k$ has a Laurent expansion

$$\wp_k(z,\tau) = \frac{1}{z^k} + (-1)^k \sum_{n=1}^{\infty} \binom{2n+1}{k-1} G_{2n+2}(\tau) z^{2n+2-k},$$

where $G_{2n+2}(\tau)$ are genuine Eisenstein series (not normalized). In parallel with $\tilde{G}_{2k}(q)$ it is convenient to introduce

$$\tilde{\wp}_k(z,q) = \frac{1}{z^k} + (-1)^k \sum_{n=1}^{\infty} \binom{2n+1}{k-1} \tilde{G}_{2n+2}(q) z^{2n+2-k}. \tag{5.1}$$

Then, for $k \geqslant 2$, we have

$$(2\pi i)^k \tilde{\wp}_k(2\pi i z, \tau) = \wp_k(z,\tau). \tag{5.2}$$

The Weierstrass ζ-function and $\wp_k$ functions are closely related to certain Jacobi modular forms. Let us fix (formal) P_{k+1}-series for $k \geqslant 1$

$$P_{k+1}(t,q) = \frac{1}{k!}\left(\sum_{n=1}^{\infty} \frac{n^k t^n}{1-q^n} + \frac{(-1)^{k+1} n^k t^{-n} q^n}{1-q^n}\right). \tag{5.3}$$

Then (5.3) converges inside $1 > |t| > |q|$ if t and q are complex variables. If we let $t = e^{2\pi i y}$, $q = e^{2\pi i \tau}$ then in the limit (cf. [19]):

$$P_2(e^{2\pi i y}, q) = \frac{1}{(2\pi i)^2}\,(\wp_2(y,\tau) + G_2(\tau)),$$

$$P_l(e^{2\pi i y}, q) = \frac{(-1)^l (l-1)!}{(2\pi i)^l}\,\wp_l(y,\tau) \tag{5.4}$$

for $l \geqslant 3$. Note that we cannot derive relation (5.4) purely by using formal variables.

The aim is to find the transformation properties of

$$\frac{1}{\mathrm{ch}_M(q)}\,\mathrm{tr}|_M X(u_1, t_1 x_1) X(v_1, x_1) X(u_2, t_2 x_2) X(v_2, x_2) q^{L(0)} \tag{5.5}$$

under the elliptic transformation:

$$t_1 \mapsto q t_1 \quad \text{and} \quad t_2 \mapsto q t_2.$$

(because of the symmetry we consider only $t_1 \mapsto q t_1$).

We will assume that (2.3) holds, i.e., that for every $i \neq j$ and $n \geqslant 0$:

$$u_i(n) v_j = c_{u_i,v_j} \delta_{\mathrm{wt}(u_i)+\mathrm{wt}(v_j)-1,n} \mathbf{1}.$$

Note that this condition implies

$$u_i[n] v_j = c_{u_i,v_j,n} \mathbf{1},$$

for every $n \geqslant 0$, where $c_{u,v,n} \in \mathbb{C}$.

Remark 5.1.

$$u(n)v = c_1 \delta_{\mathrm{wt}(u)+\mathrm{wt}(v)-1,n} \mathbf{1} \tag{5.6}$$

and

$$u[n]v = c_2 \delta_{\mathrm{wt}(u)+\mathrm{wt}(v)-1,n} \mathbf{1} \tag{5.7}$$

rarely hold simultaneously because of the different grading. More precisely, if $\mathrm{wt}(u)$ and $\mathrm{wt}(v) \geqslant 1$ then both (5.6) and (5.7) hold if and only if $\mathrm{wt}(u) = \mathrm{wt}(v) = 1$. This can be shown by writing explicitly $v[n]$ in terms of $v(i)$'s (and vice versa).

Now suppose that $\mathrm{wt}(u), \mathrm{wt}(v) \geqslant 1$ and that for every i

$$\mathrm{o}(u_i)|_M = 0, \tag{5.8}$$

The iterated 2-point function and its zeroth term is easy to compute. From the Jacobi identity for X-operators (cf. [22,24])

$$[X(u,x_1), X(v,x_2)] = \mathrm{Res}_y\, \delta\left(\frac{e^y x_2}{x_1}\right) X(Y[u,y]v, x_2)$$

it follows

$$\begin{aligned} &\mathrm{tr}|_M X(u,t_1x_1)X(v,x_1)q^{L(0)} \\ &\quad = \mathrm{tr}|_M X\left(u, \frac{t_1x_1}{q}\right) X(v,x_1)q^{L(0)} + \sum_{k\geqslant 0} \frac{c_{u,v,k}}{k!}\, \mathrm{tr}|_M D_{t_1}^k \delta(t_1) q^{L(0)}, \end{aligned} \tag{5.9}$$

where $D_{t_1} = t_1 \mathrm{d}/\mathrm{d}t_1$. Also we may assume that the summation in (5.9) goes from $k=1$.[3] Or equivalently

$$\begin{aligned} &\mathrm{tr}|_M X(u,qt_1x_1)X(v,x_1)q^{L(0)} - \mathrm{tr}|_M X(u,t_1x_1)X(v,x_1)q^{L(0)} \\ &\quad = \sum_{k=1} \frac{c_{u,v,k}}{k!}\, \mathrm{tr}|_M (D^k\delta)(qt_1)q^{L(0)}. \end{aligned}$$

Thus if we let $F(t_1) = (1/\mathrm{ch}_M(q))\, \mathrm{tr}|_M X(u_1,t_1x_1)X(v_1,x_1)q^{L(0)}$, then

$$F(qt_1) - F(t_1) = \sum_{k\geqslant 1} \frac{c_{u,v,k}}{k!} D_{t_1}^k \delta(t_1). \tag{5.10}$$

If we multiply the equation above with $(1-t_1)^{k_1+1}$ we obtain $(1-t_1)^k(F(qt_1)-F(t_1))=0$. By using Lemma A.1 from Appendix A, it follows that the general solution of (5.10) in the space

$$q^h \mathbb{C}[[t_1, t_1^{-1}, q]],$$

is of the form

$$F_{\mathrm{part}}(t_1) + f(q),$$

where $F_{\mathrm{part}}(t_1)$ is some particular solution and $f(q) \in \mathbb{C}[[q]]$. But $F(t_1)$ does not involve terms that contain only powers of q hence it follows that

$$F(t_1) = \sum_{k\geqslant 1} \frac{c_{u,v,k}}{k!} (-1)^k P_{k+1}\left(\frac{1}{t_1}\right).$$

[3] Note that for superalgebras this will not be the case.

Note that formally we have more than one solution of (5.10) inside $q^h\mathbb{C}[[t_1,t_1^{-1},q,q^{-1}]]$ (cf. Appendix A).

The 2-point functions are more interesting.

$$
\begin{aligned}
&\mathrm{tr}|_M X(u_1,t_1x_1)X(v_1,x_1)X(u_2,t_2x_2)X(v_2,x_2)q^{L(0)}\\
&\quad=\mathrm{tr}|_M X\left(u_1,\frac{t_1x_1}{q}\right)X(v_1,x_1)X(u_2,t_2x_2)X(v_2,x_2)q^{L(0)}\\
&\qquad+\mathrm{tr}|_M[X(u_1,t_1x_1),X(v_1,x_1)X(u_2,t_2x_2)X(v_2,x_2)]q^{L(0)}\\
&\quad=\mathrm{tr}|_M X\left(u_1,\frac{t_1x_1}{q}\right)X(v_1,x_1)X(u_2,t_2x_2)X(v_2,x_2)q^{L(0)}\\
&\qquad+\mathrm{tr}|_M\sum_{k_1\geqslant 1}\frac{c_{u_1,v_1,k_1}}{k_1!}D_{t_1}^{k_1}\delta(t_1)X(u_2,t_2x_2)X(v_2,x_2)q^{L(0)}\\
&\qquad+\sum_{k_2\geqslant 1}\frac{c_{u_1,u_2,k_2}}{k_2!}\mathrm{tr}|_M D_{x_1}^{k_2}\delta\left(\frac{t_1x_1}{t_2x_2}\right)X(v_1,x_1)X(v_2,x_2)q^{L(0)}\\
&\qquad+\sum_{k_3\geqslant 1}\frac{c_{u_1,v_2,k_3}}{k_3!}\mathrm{tr}|_M D_{x_1}^{k_3}\delta\left(\frac{t_1x_1}{x_2}\right)X(v_1,x_1)X(u_2,x_2t_2)q^{L(0)}\\
&\quad=\mathrm{tr}|_M X\left(u_1,\frac{t_1x_1}{q}\right)X(v_1,x_1)X(u_2,t_2x_2)X(v_2,x_2)q^{L(0)}\\
&\qquad+\sum_{k_1\geqslant 1}\frac{c_{u_1,v_1,k_1}}{k_1!}\mathrm{tr}|_M D_{t_1}^{k_1}\delta(t_1)X(u_2,t_2x_2)X(v_2,x_2)q^{L(0)}\\
&\qquad+\sum_{k_2\geqslant 1}\frac{c_{u_1,u_2,k_2}}{k_2!}\mathrm{tr}|_M\sum_{i=0}^{k_2}(-1)^i\begin{pmatrix}k_2\\ i\end{pmatrix}D_{t_1}^{k_2-i}\delta\left(\frac{t_1x_1}{t_2x_2}\right)\\
&\qquad\times D_{t_2}^{i}X(v_1,\frac{t_2x_2}{t_1})X(v_2,x_2)q^{L(0)}\\
&\qquad+\sum_{k_3\geqslant 1}c_{u_1,v_2,k_3}\mathrm{tr}|_M\sum_{i=0}^{k_3}(-1)^i\begin{pmatrix}k_3\\ i\end{pmatrix}D_{t_1}^{k_3-i}\delta\left(\frac{t_1x_1}{x_2}\right)\\
&\qquad\times X(v_1,x_1)D_{t_2}^{i}X(u_2,t_1t_2x_1)q^{L(0)},
\end{aligned}
\tag{5.11}
$$

where $k_i\in\mathbb{N}$, $i=1,2,3$. Since,

$$\mathrm{tr}|_M X(v_1,x_1)D_{t_2}^{k_3-i}X(u_2,t_1t_2x_1)q^{L(0)}=\mathrm{tr}|_M D_{t_2}^{k_3-i}X(u_2,(t_1t_2x_1/q))X(v_1,x_1)q^{L(0)},$$

we can extract zeroth term in (5.11). For simplicity suppose that $u_i=v_j=u$, $i=1,2$ and let $k_1=k_2=k_3=k$. After taking the coefficient of x^0 in (5.11) we obtain

$$F(t_1,t_2) - F\left(\frac{t_1}{q},t_2\right) = \sum_{k\geqslant 1}\frac{c_{u,u,k}}{k!}\left((D^k\delta)(t_1)F(t_2) + (-1)^k D_{t_2}^k F\left(\frac{t_2}{t_1}\right) + (-1)^k D_{t_2}^k F\left(\frac{t_1t_2}{q}\right)\right). \tag{5.12}$$

If we act by $q^{D_{t_1}}$ on (5.12) we obtain

$$F(qt_1,t_2) - F(t_1,t_2) = \sum_{k\geqslant 1}\frac{c_{u,u,k}}{k!}\left((D^k\delta)(qt_1)F(t_2) + (-1)^k D_{t_2}^k F\left(\frac{t_2}{qt_1}\right) + (-1)^k D_{t_2}^k F(t_1t_2)\right). \tag{5.13}$$

By multiplying (5.13) with $(1-t_1)^K$, where K is big enough, we obtain *locality* formula

$$(1-t_1)^K(F(qt_1,t_2) - F(t_1,t_2)) = (1-t_1)^K\sum_{k\geqslant 1}\frac{c_{u,u,k}}{k!}\left((-1)^k D_{t_2}^k F\left(\frac{t_2}{t_1q}\right) + (-1)^k D_{t_2}^k F(t_1t_2)\right). \tag{5.14}$$

The following proposition gives us an explicit formula for $F(t_1,t_2)$.

Proposition 5.1. *Suppose that*

$$H_k(qt_1,t_2) - H_k(t_1,t_2) = (D^k\delta)(qt_1)F(t_2) + D_{t_2}^k F\left(\frac{t_2}{qt_1}\right) + D_{t_2}^k F(t_1t_2), \tag{5.15}$$

where $F(t) = P_{k+1}(\frac{1}{t_1})$. *Then the general solution of* (5.15) *that satisfies* $H_k(t_1,t_2) = H_k(t_2,t_1)$ *is of the form*

(a)

$$H_k(t_1,t_2) = f_k(q) + P_{k+1}\left(\frac{1}{t_1},q\right)P_{k+1}\left(\frac{1}{t_2},q\right) - \left(\sum_{n\geqslant 1}^{\infty}\frac{n^{2k}q^n((\frac{t_1}{t_2})^n + (\frac{t_2}{t_1})^n)}{(1-q^n)^2} + \sum_{n\geqslant 1}^{\infty}\frac{n^{2k}q^n((t_1t_2)^n + (t_1t_2)^{-n})}{(1-q^n)^2}\right), \tag{5.16}$$

where $f_k(q) \in \mathbb{C}[[q]]$. *In particular if* $f(q) = 0$ *and* $|q| < 1$ *then* $H_k(t_1,t_2)$ *converges uniformly on compact subsets inside a domain*:

$$|t_1t_2q| < 1, \quad \left|\frac{q}{t_1t_2}\right| < 1, \quad \left|\frac{t_1}{t_2}q\right| < 1, \quad \left|\frac{t_2}{t_1}q\right| < 1, \quad |t_1| > 1, |t_2| > 1.$$

In particular, it has a meromorphic extension to $(\mathbb{C}^\times)^2$.

(b) *If* $t_i = e^{2\pi i y_i}$, $i=1,2$ *and* $q=e^{2\pi i\tau}$, $\tau\in\mathbb{H}$, *then*

$$H_1(\tau+y_1,y_2) = H_1(y_1,y_2) + \frac{1}{(2\pi i)^2}(\wp_2(y_1-y_2,\tau) + \wp_2(y_1+y_2,\tau) + G_2(\tau)) \tag{5.17}$$

and

$$H_k(\tau+y_1,y_2) = H_k(y_1,y_2) + \frac{1}{(2\pi i)^{k+1}}\left(\left(\frac{\partial}{\partial y_1}\right)^{k-1}\wp_2(y_1-y_2,\tau) + (-1)^{k+1}\left(\frac{\partial}{\partial y_1}\right)^{k-1}\wp_2(y_1+y_2,\tau)\right), \tag{5.18}$$

for $k\geqslant 2$, *where*

$$y_1+y_2\notin\mathbb{Z}\oplus\mathbb{Z}\tau, \quad y_1-y_2\notin\mathbb{Z}\oplus\mathbb{Z}\tau.$$

Proof. Proof of (a) requires solving (5.12), or (5.13), inside $q^h\mathbb{C}[[t_1^{\pm1},t_2^{\pm1},q]]$. The Lemma A.3 gives us description of all solutions. Hence we have to find a particular one. This can be done in the following way (cf. [7,26]). We expand the left-hand side of (5.12) in powers of t_1. Then we compare coefficients by powers t_1^n on both sides, so we can determine coefficient of t_1^n in $F(t_1,t_2)$ (since it is multiplied by $(1-q^{-n})$). The details are not very illuminated so we omit them here. This gives us a way to find a particular solution of the q-difference equation (5.12).

It is straightforward to see the uniform (and absolute) convergence of (5.16) when $f(q)=0$. If $t_i=e^{2\pi i y_i}$, (5.16) is convergent. The locality formula (5.14) gives us a meromorphic continuation of (5.16) on $(\mathbb{C}^\times)^2$ such that for every $n\in\mathbb{Z}$

$$t_1t_2\neq q^n, \quad t_1/t_2\neq q^n. \qquad \square$$

Now if we apply the Proposition 5.1 we see that $F(t_1,t_2)$ that satisfies (5.12) can be expressed as a linear combination in terms of $H_k(t_1,t_2)$, with $f_k(q)=0$.

Remark 5.2. Note that condition (2.3) is crucial for our considerations. Otherwise one cannot obtain a recursion procedure, i.e. a way to express 4-point function by using 2-point functions. Zhu's recursion (cf. [26]) would give us, in general, a way to express general 4-point function in terms of 3-point functions.

Explicit calculation of the zeroth term of (5.11) was done in [3] for $V=M(1)$, in the case when $u_i=v_i=h(-1)\mathbf{1}$, $i=1,\dots,n$, i.e.,

$$\mathscr{F}(t_1,\dots,t_n,q) = \eta(q)\,\mathrm{tr}|_{M(1)}\mathrm{o}(Y[h,t_1]h)\cdots\mathrm{o}(Y[h,t_n]h)q^{\bar{L}(0)}. \tag{5.19}$$

The approach in [3] is different since it deals with generalized characters and therefore explicit formulas are easier to obtain. In our approach we work with the formal delta function which carries an extra information not present in the generalized character formulation of (5.19).

6. Relation with vector bundles over elliptic curves

Let $\theta_{11}(t)$ be as before and $E=\mathbb{C}^\times/q^{\mathbb{Z}}$ be an elliptic curve. Suppose that $\mathscr{L}(-q^{1/2}t)\in H^0(E,\mathcal{O}_E^*)$ is a holomorphic line bundle over E. Then $\mathscr{L}(-q^{1/2}t)$ has no global section (the degree is -1). However $F(t)=1/\theta_{11}(t)$ is a multi-valued section of $\mathscr{L}(-q^{1/2}t)$.

Suppose $n=2$. Let t_1 be the local coordinate on E and t_2 fixed. If we set

$$F_1(t_1)=G(t_1,t_2)$$

and

$$F_2(t_1)=G(t_1t_2),$$

where $G(t_1,t_2)$ is Bloch–Okounkov 2-point function associated to a pair of fermions, then

$$\begin{aligned} F_1(qt_1)&=-q^{1/2}t_1t_2F_1(t_1)+q^{1/2}t_1t_2F_2(t_2),\\ F_2(qt_1)&=-q^{1/2}t_1t_2F_2(t_1). \end{aligned} \tag{6.1}$$

Therefore a pair $(F_1(t_1),F_2(t_2))$ can be though as a multi-valued section of the vector bundle $\mathscr{V}$ over E (of the rank 2) associated with the conjugacy class (which is enough to specify according to the classification theory) of the matrix

$$\begin{pmatrix} -q^{1/2}t_1t_2 & q^{1/2}t_1t_2\\ 0 & -q^{1/2}t_1t_2 \end{pmatrix}.$$

The latter corresponds to a particular (non-split) extension of $\mathscr{L}(-q^{1/2}t_1t_2)$ by itself.

This can be generalized for an arbitrary rank. Fix $t_2,\ldots,t_n$. For $1\leqslant k\leqslant n$ let

$$F_k(t_1)=\sum_{1<i_1<\cdots<i_k\leqslant n} G(t_1t_{i_1}\ldots t_{i_k},\cdots,\hat{t}_{i_1},\cdots,\hat{t}_{i_k},\cdots). \tag{6.2}$$

Notice that the right-hand side of (6.2) is invariant with respect to the symmetric group S_{n-1} (permuting $t_2,\ldots,t_n$). Then $(F_1(t_1),\ldots,F_n(t_1))$ is a multi-valued section a rank n vector bundle associated to

$$q^{1/2}\prod_{i=1}^n t_i \begin{pmatrix} -1 & 1 & -1 & 1 & \ldots & \ldots\\ 0 & -1 & \binom{2}{1} & -\binom{3}{2} & \ldots & \ldots\\ 0 & 0 & -1 & \binom{3}{1} & \ldots & \ldots\\ 0 & 0 & 0 & -1 & \ldots & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}. \tag{6.3}$$

7. Conclusion and future work

(a) Notice that one can easily generalize all results in our paper by considering more general iterated correlation functions of the form

$$\langle u'_{n+1}, X(Y[u_1, y_1 - w_1]v_1, e^{w_1}x_1) \cdots X(Y[u_n, y_n - w_n]v_n, e^{w_n}x_n)u_{n+1}\rangle,$$

$$\langle u'_{n+1}, X(u_1, t_1x_1)X(v_1, s_1x_1) \cdots X(u_n, t_nx_n)X(v_n, s_nx_n)u_{n+1}\rangle,$$

and the corresponding q-traces.

(b) Our considerations in this paper were mostly at the level of vertex operator (super)algebras and its modules. If we closely examine the n-point function $F(t_1, \ldots, t_n)$ associated to a pair of free charged fermions, we see that we are actually dealing with certain intertwining operators. More precisely, the space $\mathscr{F}_0 \cong M(1)$ is a vertex operator subalgebra inside the vertex operator superalgebra $\mathscr{F}$. Therefore operators $X(\psi, x)$ and $X(\psi^*, x)$ do not act on $M(1)$ (even though the corresponding $2n$-point trace can be computed for $M(1)$ by using the boson–fermion correspondence). Thus we are dealing with an non-trivial (even though simple) intertwining operators of the type

$$\begin{pmatrix} \mathscr{F}_0 \\ \mathscr{F}_1 \mathscr{F}_{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathscr{F}_{-1} \\ \mathscr{F}_{-1} \mathscr{F}_0 \end{pmatrix}, \tag{7.1}$$

where $\mathscr{F}_{\pm 1}$ are M-modules. The fusion algebra for $\mathscr{F}$, viewed as an intertwining operator algebra, is isomorphic to $\mathbb{C}[\mathbb{Z}]$. Therefore in the future we shall study iterated $2n$-point correlation functions and corresponding q-traces for the abelian intertwining operator algebras.

(c) It is possible to study, over the same lines, the n-point functions associated to certain $N = 2$ vertex operator superalgebras.

(d) In [24], motivated by [2], we consider a large algebra of certain pseudodifferential operators $\mathscr{D}^{\pm}_{\infty}$. In Part III [23] we study n-point functions twisted with various Dirichlet characters.

Appendix A

In this appendix we prove some elementary results necessary for solving formal q-difference equations for certain 1- and 2-point functions.

We treat the following two types of q-difference equations:

$$F(qt) - q^a F(t) = B_1(t) \tag{A.1}$$

and

$$F(qt_1, t_1) + q^{1/2} t_1 t_2 F(t_1, t_2) = B_2(t_1, t_2), \tag{A.2}$$

where $B(t) \in \mathbb{C}[[q^{\pm 1}, t^{\pm 1}]]$, $B_2(t_1, t_2) \in \mathbb{C}[[q^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}]]$. We discuss solutions both in $\mathbb{C}[[q^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}]]$ and $\mathbb{C}[[q, t_1^{\pm 1}, t_2^{\pm 1}]]$. This is important since

$$F(qt) - F(t) = D_t^k \delta(qt)$$

has two distinguished solutions in $\mathbb{C}[[q^{\pm 1}, t^{\pm 1}]]$. In general (A.1) may not have any solution. For example

$$F(qt_1) - F(t) = \delta\left(\frac{t}{q}\right),$$

does not have solution in $\mathbb{C}[[q, t_1^{\pm 1}, t_2^{\pm 1}]]$.

Remark A.1. In most of the literature one does not consider formal solutions of q-difference equations but rather solutions in some analytic spaces (like holomorphic functions in the disk, punctured disk, etc.). We work often with series with no convergence (like delta functions) thus our approach has formal-distribution theoretical flavor.

Lemma A.1. *Let $n \in \mathbb{Z}$. Then all solutions of*

$$F(qt) - q^n F(t) = 0,$$

in $\mathbb{C}[[q, t^{\pm 1}]]$ are of the form $t^n f(q)$ for some $f(q) \in \mathbb{C}[[q]]$.

Lemma A.2. *All solutions of*

$$F(qt) + q^{1/2} t F(t) = 0,$$

in $t^{1/2}\mathbb{C}[[q^{\pm 1}, t^{\pm 1}]]$ are of the form

$$f(q) \sum_{n \in \mathbb{Z}} (-1)^n q^{-n(n+1)/2} t^{n+1/2}, \tag{A.3}$$

for some $f(q) \in \mathbb{C}[[q^{\pm 1}]]$. In particular the only solution inside $t^{1/2}\mathbb{C}[[q, t^{\pm 1}]]$ is the trivial one.

Proof. Since $F(t) = \sum_{n \in \mathbb{Z}} f_{n+1/2}(q) t^{n+1/2}$, it follows that

$$q^n f_{n+1/2}(q) = -f_{n-1/2}(q).$$

Hence, $f_{1/2}(q) = -f_{-1/2}(q)$ and $f_{1/2}(q)$ uniquely determine $f_{n+1/2}(q)$. □

Lemma A.3. *All solutions of*

$$F(qt_1, t_2) - F(t_1, t_2) = 0,$$

inside $\mathbb{C}[[q^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}]]$ that satisfy $F(t_1, t_2) = F(t_2, t_1)$ are contained in $\mathbb{C}[[q^{\pm 1}]]$.

Lemma A.4. *All solutions of*

$$F(qt_1,t_2)+q^{1/2}t_1t_2F(t_1,t_2)=0, \tag{A.4}$$

in $t_1^{1/2}t_2^{1/2}\mathbb{C}[[q^{\pm1},t_1^{\pm1},t_2^{\pm1}]]$ *that satisfy*

$$F(t_1,t_2)=F(t_2,t_1) \tag{A.5}$$

are of the form

$$f(q)\sum_{n\in\mathbb{Z}}(-1)^n q^{-n(n+1)/2}(t_1t_2)^{n+1/2}.$$

In particular there is no non-trivial solution inside $\mathbb{C}[[q,t_1^{\pm1},t_2^{\pm1}]]$.

Proof. Let $F(t_1,t_2)=\sum_{n\in\mathbb{Z}+1/2}t_1^n f_n(t_2,q)$. By comparing coefficients of t_i^n in (A.4) we get

$$q^n f_n(t_2,q)+q^{1/2}t_2 f_{n-1}(t_2,q)=0,$$

On the other hand (A.5) gives us

$$f_n(qt_2,q)+q^{1/2}t_2 f_{n-1}(t_2,q).$$

Combined

$$f_n(qt_2,q)-q^n f_n(t_2,q)=0, \tag{A.6}$$

for every $n\in\mathbb{Z}+\frac{1}{2}$. Because of Lemma A.1, all solutions of (A.6), inside $t_2^{1/2}\mathbb{C}[[q,t_2^{\pm1}]]$, are of the form $t_2^n f(q)$, for some $f(q)\in\mathbb{C}[[q^{\pm1}]]$. Hence $F(t_1,t_2)=\sum_{n\in\mathbb{Z}+1/2}(t_1t_2)^n f_n(q)$. Now apply Lemma A.2. □

Appendix B.

Here we give a different proof of the main technical result used by Zhu—the recursion formula (cf. Proposition 4.3.4 of [27]) and its consequences.

Instead of working with P_{k+1}-functions and their analytic properties (5.4) we prefer the use formal variable "all the way" and then, at the end, recognize certain formal series as Laurent expansions (around $y=0$) of Weierstrass' functions. In other words, unlike Zhu's proof, our proof is completely formal-and then at the end-one can "turn on" the complex variables

First we (re)prove the following result from [26] (Proposition 4.3.2):

Proposition A.1.

$$\begin{aligned}\mathrm{tr}|_M X(u_1,x_1)X(u_2,x_2)q^{L(0)} &= \mathrm{tr}|_M \mathrm{o}(u_1)\mathrm{o}(u_2)q^{L(0)}\\ &\quad+\sum_{m\geqslant 0}P_{m+1}\left(\frac{x_2}{x_1},q\right)X(u_1[m]u_2,x_2)q^{L(0)}.\end{aligned} \tag{A.7}$$

Proof. From [22],

$$[X(u_1,x_1),X(u_2,x_2)]=\mathrm{Res}_y\,\delta\left(\frac{e^y x_2}{x_1}\right)X(Y[u_1,y]u_2,x_2).$$

Now from the property of the trace $\mathrm{tr}_M(ABC)=\mathrm{tr}|_M(BCA)$ we obtain

$$(1-q^{-D_{x_1}})X(u_1,x_1)X(u_2,x_2)q^{L(0)}=\sum_{m\geqslant 0}\frac{D^m}{m!}\,\delta\left(\frac{x_2}{x_1}\right)X(u_1[m]u_2,x_2). \tag{A.8}$$

Now, by formally "inverting" the operator $1-q^{-D_{x_1}}$ inside $\mathbb{C}[[q]]$ we obtain the formula (A.7). $\square$

Let us recall Weierstrass function $\wp_k(z,\tau)$ defined before. These functions have a Laurent expansion near $z=0$ $(k=1,2,\ldots)$:

$$\wp_k(z,\tau)=\frac{1}{z^k}+(-1)^k\sum_{n\geqslant 1}\binom{2n+1}{k-1}G_{2n+2}(\tau)z^{2n+2-k}. \tag{A.9}$$

Notice that (A.9) can be considered formally as an element of

$$\mathbb{C}[[z,z^{-1},q]].$$

Let us define related function:

$$\bar{\wp}_k(z,q)=\begin{cases}\tilde{\wp}_2(z,q)+\tilde{G}_2(q) & \text{for } k=2\\ \tilde{\wp}_k(z,q), & k\geqslant 3\end{cases} \tag{A.10}$$

Theorem A.1.

$$\begin{aligned}&\mathrm{tr}|_M X(Y[u,y]v,x)\\ &\quad=\sum_{m\geqslant 1}\bar{\wp}_{m+1}(y)\,\mathrm{tr}|_M X(u[m]v,x)q^{L(0)}+\mathrm{tr}|_M\,\mathrm{o}(u)\mathrm{o}(v)q^{L(0)}.\end{aligned} \tag{A.11}$$

In particular, if M is a V-module and V satisfies C_2-condition (cf. [26]) then (A.1) is convergent and it has a meromorphic continuation to the whole y plane.

Here are some consequences (cf. Propositions 4.3.4 and 4.3.5 in [27]):

Corollary A.1.

$$\begin{aligned}\mathrm{tr}|_M\,\mathrm{o}(u[-1]v)q^{L(0)}&=\mathrm{tr}|_M\,\mathrm{o}(u)\mathrm{o}(v)q^{L(0)}\\ &\quad+\sum_{m\geqslant 1}G_{2k}(q)\,\mathrm{tr}|_M\,\mathrm{o}(u[2k-1]v)q^{L(0)}.\end{aligned} \tag{A.12}$$

Combining Corollary A.1 and Proposition A.1 we obtain

Corollary A.2.

$$\begin{aligned}
&\mathrm{tr}|_M X(u,x_1)X(v,x_2)q^{L(0)} \\
&\quad = \mathrm{tr}|_M \mathrm{o}(u[-1]v)q^{L(0)} + \frac{1}{2}\,\mathrm{tr}|_M \mathrm{o}(u[0]v)q^{L(0)} \\
&\qquad + \mathrm{tr}|_M \sum_{m\geqslant 0} P_{m+1}\left(\frac{x_2}{x_1},q\right)\mathrm{o}(u[m]v)q^{L(0)} - \sum_{k\geqslant 1}\mathrm{tr}|_M G_{2k}(q)\mathrm{o}(u[2k-1]v)q^{L(0)}.
\end{aligned} \tag{A.13}$$

Corollary A.3. *Let* $q=e^{2\pi i\tau}$. *Expression* (A.1) *is a doubly periodic* (*elliptic*) *function with respect to transformations*

$$y \mapsto y+2\pi i,$$

$$y \mapsto y+2\pi i\tau.$$

Proof of Theorem A.1.

$$\begin{aligned}
&\mathrm{tr}|_M X(Y[u,y]v,x_2)q^{L(0)} \\
&\quad = \sum_{i\in\mathbb{Z}}\mathrm{tr}|_M X(u(i)v,x_2)e^{y\deg(u)}(e^y-1)^{-i-1}q^{L(0)} \\
&\quad = \mathrm{Res}_{x_1}\sum_{i\in\mathbb{Z}} x_2^{\deg(u)+\deg(v)-i-1}e^{y\deg(u)}(e^y-1)^{-i-1} \\
&\qquad \times \mathrm{tr}|_M((x_1-x_2)^i Y(u,x_1)Y(v,x_2)-(-x_2+x_1)^i Y(v,x_2)Y(u,x_1))q^{L(0)} \\
&\quad = \mathrm{Res}_{x_1}\sum_{i\in\mathbb{Z}} e^{y\deg(u)}(e^y-1)^{-i-1}x_2^{-1}\left(\frac{x_2}{x_1}\right)^{\deg(u)} \\
&\qquad \times \mathrm{tr}|_M(x_2^{-i}(x_1-x_2)^i X(u,x_1)X(v,x_2)-\ x_2^{-i}(-x_2+x_1)^i X(v,x_2)X(u,x_1))q^{L(0)} \\
&\quad = \mathrm{Res}_{x_1}\sum_{i\in\mathbb{Z}} e^{y\deg(u)}(e^y-1)^{-i-1}x_2^{-1}\left(\frac{x_2}{x_1}\right)^{\deg(u)} \\
&\qquad \times \left(x_2^{-i}(x_1-x_2)^i\sum_{m\geqslant 0}\mathrm{tr}|_M P_{m+1}\left(\frac{x_2}{x_1},q\right)\mathrm{o}(u[m]v)q^{L(0)}\right. \\
&\qquad \left. -x_2^{-i}(-x_2+x_1)^i\sum_{m\geqslant 0}\mathrm{tr}|_M P_{m+1}\left(\frac{qx_2}{x_1},q\right)\mathrm{o}(u[m]v)q^{L(0)}\right)
\end{aligned}$$

$$+\operatorname{Res}_{x_1}\sum_{i\in\mathbb{Z}} e^{y\deg(u)}(e^y-1)^{-i-1}x_2^{-1}\left(\frac{x_2}{x_1}\right)^{\deg(u)}$$
$$\times\left\{(x_2^{-i}(x_1-x_2)^i - x_2^{-i}(-x_2+x_1)^i)\operatorname{tr}|_M \mathrm{o}(a)\mathrm{o}(b)q^{L(0)}\right\}. \tag{A.14}$$

We introduce a substitution $t=x_1/x_2$. Then

$$\operatorname{Res}_{x_1}\sum_{i\in\mathbb{Z}} e^{y\deg(u)}(e^y-1)^{-i-1}x_2^{-1}\left(\frac{x_2}{x_1}\right)^{\deg(u)}$$
$$\times\left\{(x_2^{-i}(x_1-x_2)^i - x_2^{-i}(-x_2+x_1)^i)\operatorname{tr}|_M \mathrm{o}(a)\mathrm{o}(b)q^{L(0)}\right\}$$
$$=\operatorname{Res}_t\sum_{i\in\mathbb{Z}} e^{y\deg(u)}(e^y-1)^{-i-1}t^{-\deg(u)}((t-1)^i$$
$$-(-1+t)^i)\operatorname{tr}|_M \mathrm{o}(a)\mathrm{o}(b)q^{L(0)}$$
$$=\operatorname{Res}_t\sum_{i\leqslant -1} e^{y\deg(u)}(e^y-1)^{-i-1}t^{-\deg(u)}(t-1)^i\operatorname{tr}|_M \mathrm{o}(a)\mathrm{o}(b)q^{L(0)}$$
$$-\operatorname{Res}_t\sum_{i\leqslant -1}(-1+t)^i e^{y\deg(u)}(e^y-1)^{-i-1}t^{-\deg(u)}\operatorname{tr}|_M \mathrm{o}(a)\mathrm{o}(b)q^{L(0)}$$
$$=\operatorname{Res}_t\sum_{i\geqslant 0} e^{y\deg(u)}\frac{(e^y-1)^i}{(t-1)^{i+1}}t^{-\deg(u)}\operatorname{tr}|_M \mathrm{o}(a)\mathrm{o}(b)q^{L(0)}$$
$$-\operatorname{Res}_t\sum_{i\geqslant 0} e^{y\deg(u)}\frac{(e^y-1)^i}{(-1+t)^{i+1}}t^{-\deg(u)}\operatorname{tr}|_M \mathrm{o}(a)\mathrm{o}(b)q^{L(0)}$$
$$=\operatorname{Res}_t\left(e^{y\deg(u)}t^{-\deg(u)-1}\frac{1}{1-e^y/t}\operatorname{tr}|_M \mathrm{o}(a)\mathrm{o}(b)q^{L(0)}\right)$$
$$+\operatorname{Res}_t\left(t^{-\deg(u)}e^{y\deg(u)-1}\frac{1}{1-t/e^y}\operatorname{tr}|_M \mathrm{o}(a)\mathrm{o}(b)q^{L(0)}\right)$$
$$=\operatorname{tr}|_M \mathrm{o}(a)\mathrm{o}(b)q^{L(0)}. \tag{A.15}$$

Also for every $m\geqslant 1$

$$\operatorname{Res}_{x_1}\sum_{i\in\mathbb{Z}} e^{y\deg(u)}(e^y-1)^{-i-1}x_2^{-1}\left(\frac{x_2}{x_1}\right)^{\deg(u)}$$
$$\times\left\{x_2^{-i}(x_1-x_2)^i P_{m+1}\left(\frac{x_2}{x_1},q\right) - x_2^{-i}(-x_2+x_1)^i P_{m+1}\left(\frac{qx_2}{x_1},q\right)\right\}$$

$$
\begin{aligned}
&= \mathrm{Res}_t \left\{ \sum_{i\in\mathbb{Z}} e^{y\deg(u)} (e^y-1)^{-i-1} x_2^{-1} t^{-\deg(u)} \right.\\
&\quad \left. \times \{(t-1)^i P_{m+1}(t^{-1},q) - (-1+t)^i P_{m+1}(qt^{-1},q)\} \right\}\\
&= \mathrm{Res}_t \left\{ e^{y\deg(u)} t^{-\deg(u)-1} \frac{1}{1-e^y/t} P_m(t^{-1},q) \right.\\
&\quad \left. + t^{-\deg(u)} e^{y\deg(u)-1} \frac{1}{1-t/e^y} P_m(qt^{-1},q) \right\}\\
&\quad + \mathrm{Res}_t \left\{ \frac{e^{y\deg(u)}}{e^y-1} t^{\deg(u)} \sum_{i\geqslant 0} \frac{(t^{-1}-1)^i}{(e^y-1)^i} \frac{D^{m-1}}{(m-1)!} \delta(t) \right\},
\end{aligned}
\tag{A.16}
$$

where we used the fact that

$$
P_m(t,q) - P_m(qt,q) = \frac{D^{m-1}}{(m-1)!} \delta(t).
$$

If we combine all P_m's into a single generating function we obtain

$$
\sum_{m\geqslant 1} (P_m(t,q) - P_m(qt,q)) x^{m-1} = \delta(e^x t).
$$

Now

$$
\begin{aligned}
&\mathrm{Res}_t \frac{e^{y\deg(u)}}{e^y-1} t^{\deg(u)} \sum_{i=0}^{\infty} \frac{(t^{-1}-1)^i}{(e^y-1)^i} \delta(e^x t)\\
&= \mathrm{Res}_t \frac{e^{y\deg(u)}}{e^y-1} e^{-x\deg(u)} \sum_{i=0}^{\infty} \frac{(e^x-1)^i}{(e^y-1)^i} \delta(e^x t) = \frac{e^{(y-x)\deg(u)}}{e^{y-x}-1}.
\end{aligned}
\tag{A.17}
$$

In order to evaluate the last sum in (A.16) we have to extract the coefficient of x^{m-1} inside (A.17). Clearly

$$
\begin{aligned}
&\mathrm{Coeff}_{x^{m-1}} \frac{e^{(y-x)\deg(u)}}{e^{y-x}-1} = \mathrm{Coeff}_{x^{m-1}} e^{-x\mathrm{d}/\mathrm{d}y} \frac{e^{y\deg(u)}}{e^y-1}\\
&= \frac{(-1)^{m-1}}{(m-1)!} \left(\frac{\partial}{\partial y}\right)^{m-1} \sum_{k=0}^{\infty} \frac{B_k(\deg(u)) y^{k-1}}{k!}.
\end{aligned}
\tag{A.18}
$$

Now, by using the formula

$$
P_m(t,q) = \frac{1}{(m-1)!} \left(\sum_{n\geqslant 1} \frac{n^{m-1} t^n}{1-q^n} + (-1)^m \sum_{n\geqslant 1} \frac{n^{m-1} q^n t^{-n}}{1-q^n} \right),
$$

we obtain

$$
\begin{aligned}
&\mathrm{Res}_t \left\{ e^{y\deg(u)} t^{-\deg(u)-1} \frac{1}{1-e^y/t} P_m(t^{-1},q) \right.\\
&\quad \left. + t^{-\deg(u)} e^{y\deg(u)-1} \frac{1}{1-t/e^y} P_m(qt^{-1},q) \right\}\\
&\quad = \frac{(-1)^{m-1}}{(m-1)!} \sum_{n \geqslant 0} e^{y(\deg(u)+n)} \frac{q^{\deg(u)+n}(\deg(u)+n)^{m-1}}{1-q^{\deg(u)+n}}\\
&\quad + \frac{1}{(m-1)!} \sum_{n} e^{-yn} \frac{n^{m-1}q^n}{1-q^n} + \frac{(-1)^m}{(m-1)!} \sum_{n=1}^{\deg(u)-1} e^{yn} \frac{n^{m-1}q^n}{1-q^n}.
\end{aligned}
\tag{A.19}
$$

From (A.18) and (A.19) it follows ($m \geqslant 2$):

$$
\begin{aligned}
&\mathrm{Res}_{x_1} \sum_{i \in \mathbb{Z}} e^{y\deg(u)} (e^y-1)^{-i-1} x_2^{-1} \left(\frac{x_2}{x_1}\right)^{\deg(u)} \left\{ \left(x_2^{-i}(x_1-x_2)^i P_m\left(\frac{x_2}{x_1},q\right) \right.\right.\\
&\quad \left.\left. - x_2^{-i}(-x_2+x_1)^i P_m\left(\frac{qx_2}{x_1},q\right)\right) \mathrm{tr}|_M \mathrm{o}(u[m-1]v)q^{L(0)} \right\}\\
&\quad = \left\{ \frac{1}{y^m} + \frac{(-1)^{m-1}}{(m-1)!} \sum_{n \geqslant 0} \frac{B_{m+n}(\deg(u))y^n}{n!} + \frac{(-1)^m}{(m-1)!} \sum_{n=1}^{\deg(u)-1} e^{yn} n^{m-1} \right.\\
&\quad \left. + \frac{(-1)^m}{(m-1)!} \left(\sum_{n \geqslant 1} e^{yn} \frac{n^{m-1}q^n}{1-q^n} + (-1)^m e^{-yn} \frac{n^{m-1}q^n}{1-q^n} \right) \right\}\\
&\quad \times \mathrm{tr}|_M \mathrm{o}(u[m-1]v)q^{L(0)}\\
&\quad = \left\{ \frac{1}{y^m} + \frac{(-1)^m}{(m-1)!} \sum_{r \geqslant 0} \left(\frac{B_{m+r}(\deg(u))}{m+r} - \frac{B_{m+r}}{m+r} \right) \frac{y^r}{r!} \right.\\
&\quad \left. + \frac{(-1)^m}{(m-1)!} \left(\sum_{r=0}^{\infty} \sum_{n=1}^{\infty} \frac{n^{r+m-1}q^n y^r}{r!(1-q^n)} + \frac{(-1)^{m+r} n^{r+m-1} q^n y^r}{r!(1-q^n)} \right) \right\}\\
&\quad \times \mathrm{tr}|_M \mathrm{o}(u[m-1]v)q^{L(0)}\\
&\quad = \left\{ \frac{1}{y^m} + \frac{(-1)^m}{(m-1)!} \sum_{r \geqslant 0, r+m \in 2\mathbb{Z}} \left(-\frac{B_{m+r}y^r}{(m+r)r!} \right.\right.\\
&\quad \left.\left. + \frac{2}{(r+m-1)!} \sum_{n=1}^{\infty} \binom{r+m-1}{m-1} \frac{n^{m+r-1}q^n}{1-q^n} \right) y^r \right\}
\end{aligned}
$$

$$
\begin{aligned}
&\times \mathrm{tr}|_M \mathrm{o}(u[m-1]v)q^{L(0)}\\
&= \left\{ \frac{1}{y^m} + \frac{(-1)^m}{(m-1)!} \sum_{k \geqslant m/2, k\in\mathbb{N}} \binom{2k-1}{m-1} \right.\\
&\quad \left. \times \left(\frac{-B_{2k}}{(2k)!} + \frac{2}{(2k-1)!} \sum_{n=1}^{\infty} \frac{n^{2k-1}q^n}{1-q^n} \right) y^{2k-m} \right\}\\
&\quad \times \mathrm{tr}|_M \mathrm{o}(u[m-1]v)q^{L(0)}\\
&= \left(\frac{1}{y^m} + (-1)^m \sum_{k \geqslant m/2, k\in\mathbb{N}} \binom{2k-1}{m-1} \tilde{G}_{2k}(q) y^{2k-m} \right) \mathrm{tr}|_M \mathrm{o}(u[m-1]v)q^{L(0)}\\
&= \begin{cases} (\tilde{\wp}_2(y,q) + \tilde{G}_2(q))\, \mathrm{tr}|_M \mathrm{o}(u[1]v)q^{L(0)} & \text{for } m=2\\ \tilde{\wp}_m(y,q)\, \mathrm{tr}|_M \mathrm{o}(u[m-1]v)q^{L(0)} & \text{for } m>2. \end{cases}\\
&= \bar{\wp}_m(y,q)\, \mathrm{tr}|_M \mathrm{o}(u[m-1]v)q^{L(0)}.
\end{aligned} \tag{A.20}
$$

We used a fact that

$$\tilde{G}_{2k}(\tau) = -\frac{B_{2k}}{(2k)!} + \frac{2}{(2k-1)!} \sum_{n=1}^{\infty} \frac{n^{2k-1}q^n}{1-q^n}.$$

Because of

$$\mathrm{tr}|_M \mathrm{o}(u[0]v)q^{L(0)} = 0, \tag{A.21}$$

formulas (A.15), (A.20) and (A.21) imply

$$
\begin{aligned}
&\mathrm{tr}|_M X(Y[u,y]v,x)q^{L(0)}\\
&\quad = \sum_{m\geqslant 1} \bar{\wp}_{m+1}(y,q)\, \mathrm{tr}|_M X(u[m]v,x)q^{L(0)} + \mathrm{tr}|_M \mathrm{o}(u)\mathrm{o}(v)q^{L(0)}. \qquad \square
\end{aligned} \tag{A.22}
$$

Proof of Corollary A.3. If we let $q = e^{2\pi i\tau}$, the ellipticity with respect to

$$y \mapsto y + 2\pi i,$$

$$y \mapsto y + 2\pi i\tau$$

follows directly from formulas (5.1) and (5.2). □

References

[1] H. Awata, M. Fukuma, Y. Matsuo, S. Odake, Representation theory of the $W_{1+\infty}$ algebra, Progr. Theoret. Phys. (Suppl.) 118 (1995) 343–373, Quantum field theory, integrable models and beyond, Kyoto, 1994.

[2] S. Bloch, Zeta values and differential operators on the circle, J. Algebra 182 (2) (1996) 476–500.

[3] S. Bloch, A. Okounkov, The character of the infinite wedge representation, Adv. Math. 149 (1) (2000) 1–60.
[4] R. Dijkgraaf, Mirror symmetry and elliptic curves, in: The Moduli Space of Curves (Texel Island, 1994), Birkhäuser Boston, Boston, MA, 1995, pp. 149–163.
[5] R. Dijkgraaf, Chiral deformations of conformal field theories, Nuclear Phys. B 493 (3) (1997) 588–612.
[6] C. Dong, J. Lepowsky, Generalized Vertex Algebras and Relative Vertex Operators, Birkhäuser Boston Inc., Boston, MA, 1993.
[7] G. Felder, R. Silvotti, Modular covariance of minimal model correlation functions, Comm. Math. Phys. 123 (1) (1989) 1–15.
[8] E. Frenkel, V. Kac, A. Radul, W. Wang, $\mathscr{W}_{1+\infty}$ and $\mathscr{W}(gl_N)$ with central charge, N. Comm. Math. Phys. 170 (2) (1995) 337–357.
[9] I. Frenkel, Y.-Z. Huang, J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, Mem. Amer. Math. Soc. 104 (1993) 494.
[10] I. Frenkel, J. Lepowsky, A. Meurman, Vertex Operator Algebras and the Monster, Academic Press Inc., Boston, MA, 1988.
[11] Y.-Z. Huang, Two-dimensional Conformal Geometry and Vertex Operator Algebras, Birkhäuser Boston Inc., Boston, MA, 1997.
[12] V.G. Kac, Vertex Algebras for Beginners, American Mathematical Society, Providence, RI, 2nd Edition, 1998.
[13] V.G. Kac, D.H. Peterson, Spin and wedge representations of infinite-dimensional Lie algebras and groups, Proc. Nat. Acad. Sci. U.S.A. 78 (6, Part 1) (1981) 3308–3312.
[14] V.G. Kac, A. Radul, Quasifinite highest weight modules over the Lie algebra of differential operators on the circle, Comm. Math. Phys. 157 (3) (1993) 429–457.
[15] V.G. Kac, M. Wakimoto, Integrable highest weight modules over affine superalgebras and appell's function, Comm. Math. Phys. 215 (2001) 631–682.
[16] V.G. Kac, W. Wang, Vertex operator superalgebras and their representations, in: Mathematical Aspects of Conformal and Topological Field Theories and Quantum Groups (South Hadley, MA, 1992), Amer. Math. Soc., Providence, RI, 1994, pp. 161–191.
[17] V.G. Kac, W. Wang, C.H. Yan, Quasifinite representations of classical Lie subalgebras of $\mathscr{W}_{1+\infty}$, Adv. Math. 139 (1) (1998) 56–140.
[18] M. Kaneko, D. Zagier, A generalized Jacobi theta function and quasimodular forms, in: The Moduli Space of Curves (Texel Island, 1994), Birkhäuser Boston, Boston, MA, 1995, pp. 165–172.
[19] S. Lang, Elliptic Functions, Springer, New York, 2nd Edition, 1987 (With an appendix by J. Tate).
[20] J. Lepowsky, to appear.
[21] J. Lepowsky, Vertex operator algebras and the zeta function, in: Recent Developments in Quantum Affine Algebras and Related Topics (Raleigh, NC, 1998), Amer. Math. Soc., Providence, RI, 1999, pp. 327–340.
[22] J. Lepowsky, Application of a "Jacobi identity" for vertex operator algebras to zeta values and differential operators, Lett. Math. Phys. 53 (2) (2000) 87–103.
[23] A. Milas, Formal Differential Operators, Vertex Operator Algebras and Zeta-functions, III, to appear.
[24] A. Milas, Formal Differential Operators, Vertex Operator Algebras and Zeta-values, I, to appear in J. Pure Appl. Algebra.
[25] H. Tsukada, String path integral realization of vertex operator algebras, Mem. Amer. Math. Soc. 91 (444) (1991).
[26] Y. C. Zhu, Vertex operators, elliptic functions and modular forms, Ph.D. Thesis, Yale University, 1990.
[27] Y.C. Zhu, Modular invariance of characters of vertex operator algebras, J. Amer. Math. Soc. 9 (1) (1996) 237–302.